Integr. equ. oper. theory 99 (9999) 0–0
0378-620X/99000-0 $ 1.50+0.20/0

Integral Equations and Operator Theory

Solvability and Fredholm properties of integral equations on the half-line in weighted spaces

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Abstract. The solvability of integral equations of the form $\lambda x(s) = y(s) + \int_0^\infty k(s,t)x(t)\,dt$ and the behaviour of the solution x at infinity are investigated. Conditions on k and on a weight function w are obtained which ensure that the integral operator K with kernel k is bounded as an operator on X_w, where X_w denotes the weighted space of those continuous functions defined on the half-line which are $O(w(s))$ as $s \to \infty$. We also derive conditions on w and k which imply that the spectrum and essential spectrum of K on X_w are the same as on $BC[0,\infty)$. In particular, the results apply when $k(s,t) = \kappa(s-t)$, $\kappa \in L^1(\mathbb{R})$, when the integral equation is of Wiener-Hopf type. In this case we show that our results are particularly sharp.

Mathematics Subject Classification (2000). Primary 45B05; Secondary 42A85, 45M05, 47A10, 47G10.

Keywords. Wiener-Hopf integral equation, half-line, weighted space, spectrum, essential spectrum.

1. Introduction

In this paper we consider integral equations on the half-line of the form

$$\lambda x(s) - \int_0^\infty k(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}_+, \tag{1.1}$$

where the given right hand side y and the sought solution x belong to the space X of bounded and continuous functions on $\mathbb{R}_+ := [0,\infty)$. We require that the kernel function $k : \mathbb{R}_+^2 \to \mathbb{C}$ satisfies $k_s := k(s,\cdot) \in L^1(\mathbb{R}_+)$ for all $s \in \mathbb{R}_+$, so that the integral in (1.1) exists in a Lebesgue sense for every $s \in \mathbb{R}_+$.

We define the integral operator K by

$$Kx(s) = \int_0^\infty k(s,t)x(t)\,dt, \quad s \in \mathbb{R}_+, \tag{1.2}$$

so that we can abbreviate (1.1) as

$$\lambda x - Kx = y.$$

Throughout the paper we also assume that the kernel k satisfies the following two conditions:

$$\boldsymbol{(A)} \qquad \sup_{s\in\mathbb{R}_+} ||k_s||_1 = \sup_{s\in\mathbb{R}_+} \int_0^\infty |k(s,t)|\,dt < \infty,$$

and

$$\boldsymbol{(B)} \qquad \forall s \in \mathbb{R}_+ \int_0^\infty |k(s,t) - k(s',t)|\,dt \to 0 \quad \text{as } s' \to s.$$

It is well known that conditions $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$ ensure that K is a bounded operator on X with operator norm

$$||K|| = \sup_{s\in\mathbb{R}_+} ||k_s||_1. \tag{1.3}$$

The main aim of this paper is to relate the solvability of (1.1) in X to its solvability in weighted spaces of continuous functions. Our assumption throughout is that the weight function $w \in C(\mathbb{R}_+)$ satisfies that

$$w(0) = 1, \quad w(s) \geq w(t) \text{ for } s \geq t \geq 0, \quad \lim_{s\to\infty} w(s) = \infty. \tag{1.4}$$

We denote by X_w the subspace of X consisting of all functions $x \in X$ satisfying that $x(s) = O(1/w(s))$ as $s \to \infty$. X and X_w are Banach spaces with the norms

$$||x|| := \sup_{s\in\mathbb{R}_+} |x(s)|, \qquad ||x||_w := ||xw||,$$

respectively.

We start our investigation by deriving sufficient conditions on k which ensure that $K : X_w \to X_w$ and is bounded, i.e. that $K \in B(X_w)$, where $B(X_w)$ is the set of bounded linear operators on X_w. For weight functions which satisfy the condition

$$\frac{w(s+1)}{w(s)} \to 1, \quad \text{as } s \to \infty, \tag{1.5}$$

and kernels for which the bound

$$\boldsymbol{(A')} \quad |k(s,t)| \leq |\kappa(s-t)|, \quad s,t \in \mathbb{R}_+, \tag{1.6}$$

for some $\kappa \in L^1(\mathbb{R})$ holds, we establish general conditions on κ and w which imply the much stronger result that the spectrum of K is the same on X as on X_w, in symbols,

$$\Sigma_X(K) = \Sigma_{X_w}(K). \tag{1.7}$$

The same conditions imply that the essential spectrum is the same on X as on X_w (see, e.g., [15] for the definitions of Fredholm theory), in symbols,

$$\Sigma^e_X(K) = \Sigma^e_{X_w}(K). \tag{1.8}$$

With regard to the integral equation (1.1), of course the spectral equivalence (1.7) implies that the equation (1.1) has a unique solution $x \in X$ for every $y \in X$ if and only if it has a unique solution $x \in X_w$ for every $y \in X_w$.

Note that the condition (1.5) limits the growth of w, implying that

$$w(s) = o(e^{bs}), \quad \text{as } s \to \infty,$$

for every $b > 0$. At the end of Section 2 we exhibit examples which show that if $w(s) = e^{bs}$ for some $b > 0$, then (1.7) and (1.8) may or may not hold. We also show that, if $k(s,t) = 0$ for $0 \le t \le s$, then $\Sigma_{X_w}(K) \subset \Sigma_X(K)$ holds for every w satisfying (1.4), though an example for the case $w(s) = e^{bs}$ shows that neither (1.7) nor $\Sigma^e_{X_w}(K) \subset \Sigma^e_X(K)$ need hold.

The results described above comprise Section 2 of the present paper. In Section 3 we give various sufficient conditions on kernels and weight functions which ensure that the assumptions of our main theorem on the equivalence of spectra are fulfilled. These conditions are easier to check in applications than our main assumptions in Section 2. Further, we provide and discuss examples of kernels and weight functions to which our results apply. At the end of Section 3 we show, given any kernel satisfying $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$, with $\kappa \in L^1(\mathbb{R})$, how to construct a weight function w satisfying (1.4) such that (1.7) and (1.8) hold. As an application of this result we establish that $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$ imply that $\Sigma_{X_0}(K) \subset \Sigma_X(K)$, where X_0 is the closed subspace of X consisting of those $\phi \in X$ which vanish at infinity.

A special case of some interest is that when $k(s,t) = \kappa(s-t)$ for some $\kappa \in L^1(\mathbb{R})$, in which case (1.1) is the integral equation of Wiener-Hopf type

$$\lambda x(s) - \int_0^\infty \kappa(s-t)x(t)\,dt = y(s), \quad s \in \mathbb{R}_+. \tag{1.9}$$

It is well known that then $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$ are satisfied with $\sup_{s\in\mathbb{R}_+} \|k_s\|_1 = \|\kappa\|_1$. In this Wiener-Hopf case the conditions we impose on k to obtain that $K \in B(X_w)$ and the main results (1.7) and (1.8) are, for many weight functions w, both necessary and sufficient. For example, consider the particular weight function

$$w(s) = \exp(as^\alpha)(1+s)^p\big(\ln(e+s)\big)^q, \quad s \in \mathbb{R}_+, \tag{1.10}$$

and suppose that the constants $\alpha \in (0,1)$, $a \ge 0, p, q \in \mathbb{R}$ are such that (1.4) holds and $\int_0^\infty w^{-1}(s)\,ds$ is finite. Then the results we obtain imply for the Wiener-Hopf case $k(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, that a necessary and sufficient condition for $K \in B(X_w)$ is

$$w(s)\int_s^{s+1} |\kappa(t)|\,dt = O(1), \quad s \to \infty. \tag{1.11}$$

Moreover this condition ensures the spectral equalities (1.7) and (1.8) hold. In the more general case that the kernel k satisfies $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$ with $\kappa \in L^1(\mathbb{R})$, it remains true that (1.11) also ensures that $K \in B(X_w)$ and (1.7) and (1.8) hold.

Our study continues earlier investigations in [14] (see also the monograph [15] and [18]) which consider primarily the case $w(s) = (1+s)^r$ for some $r \in \mathbb{R}$. In

[15, 14, 18] it is shown, using Banach algebra techniques, that in the Wiener-Hopf case $k(s,t) = \kappa(s-t)$ it holds that $K \in B(X_w)$ if

$$\int_{-\infty}^{\infty} (1+|t|)^r |\kappa(t)| \, dt < \infty \tag{1.12}$$

and that if (1.12) holds then

$$\Sigma^e_{X_w}(K) = \Sigma^e_X(K) = \{\hat{\kappa}(\xi) : \xi \in \mathbb{R}\} \cup \{0\} \tag{1.13}$$

and

$$\Sigma_{X_w}(K) = \Sigma_X(K) = \Sigma^e_X(K) \cup \{\lambda \in \mathbb{C} : [\arg(\lambda - \hat{\kappa}(\xi))]^{\infty}_{-\infty} \neq 0\}, \tag{1.14}$$

where

$$\hat{\kappa}(\xi) = \int_{-\infty}^{\infty} \kappa(t) e^{i\xi t} \, dt, \tag{1.15}$$

is the Fourier transform of κ. The explicit expressions (1.13) and (1.14), for the essential spectrum and spectrum, date back to Krein [13] where it is shown that these formulae specify the essential spectrum and spectrum of the Wiener-Hopf integral operator K as an operator on X, X_0, and $L^p(\mathbb{R}_+)$, $1 \le p \le \infty$.

In [12] (see also [11]) Karapetiants and Samko provide results for convolution kernels which include the result of [14] as a special case, based on a demonstration that $K - K_w$ is compact on X, where K_w is the integral operator of the form (1.2) with kernel

$$k_w(s,t) := \frac{w(s)}{w(t)} \kappa(s-t), \quad s,t \in \mathbb{R}_+. \tag{1.16}$$

Indeed, all the spectra and, respectivly, essential spectra in the previous paragraph coincide, bearing in mind a well-known general result of Krein [13]: for every $\kappa \in L^1(\mathbb{R})$ and $1 \le p \le \infty$ there hold

$$\Sigma_X(K) = \Sigma_{X_0}(K) = \Sigma_{L^p(\mathbb{R}_+)},$$
$$\Sigma^e_X(K) = \Sigma^e_{X_0}(K) = \Sigma^e_{L^p(\mathbb{R}_+)}.$$

The more general case when k is not a convolution kernel has received little explicit attention in the literature. But, in a series of papers [4, 5, 3] the case when k satisfies $\boldsymbol{(A')}$ is considered, with $w(s) = (1+s)^p$ for some $p > 0$ (so that w satisfies the conditions (1.4)). It is shown that if

$$\kappa(s) = O(s^{-q}), \quad s \to \infty, \tag{1.17}$$

for some $q > 1$ then $K \in B(X_w)$ and (1.7) and (1.8) hold for $0 < p \le q$. A key component of the argument is the consideration, as in Samko [12], of properties of $K - K_w$. In the limiting case when $p = q$, $K - K_w$ may not be compact but is a sufficiently well-behaved operator (see Section 2 below) to proceed by somewhat similar arguments to the case when $K - K_w$ is compact. We point out that for many applications the condition that (1.17) holds for some $q > 1$ with $q \ge p$ is a much less onerous condition than (1.12). In particular, in the case that $|\kappa(s)| \sim as^{-q}$ as $s \to \infty$, for some $a > 0$, in which case necessarily $q > 1$ given that $\kappa \in L^1(\mathbb{R})$, the results of [15] and [12] give that $K \in B(X_w)$ and (1.7) and (1.8) hold for $0 < p \le q$, while (1.12) holds with $r = p$, and so the theory of [15] and [12] applies only if $|p| < q - 1$.

The present paper can be considered in large part as an attempt to sharpen and generalize the results and methods of argument of [4, 5, 3], establishing large classes of kernels k and weight functions w for which $K \in B(X_w)$ and (1.7) and (1.8) hold. The special case referred to above for the weight function (1.10) contains many of the results of [4, 5, 3]. For the weight $w(s) = (1+s)^r$ with $r > 1$ and the Wiener-Hopf case $k(s,t) = \kappa(s-t)$, the general results of this paper show that $K \in B(X_w)$ if and only if

$$\int_s^{s+1} |\kappa(t)|\, dt = O(s^{-r}), \quad s \to \infty. \tag{1.18}$$

This condition does not imply that $K - K_w$ is compact but, nevertheless, ensures that (1.7) and (1.8) hold. Note that (1.18) is a considerably weaker condition than (1.12).

Throughout Sections 2 and 3 we restrict our attention to the case when K is an integral operator on the half-line. We expect, based on our experience with the power weight $w(s) = (1+s)^r$ [5, 8], that it should be possible to extend our results and arguments to integral equations on more general multidimensional unbounded domains or systems of such equations.

In Section 4 we briefly explain how our assumptions and results can be modified to apply to integral operators on the real line. The results presented in this section have recently found an important application in the analysis of the finite section method for integral equations on the real line of the form

$$x(s) - \int_{-\infty}^{\infty} \kappa(s-t) z(t) x(t)\, dt = y(s), \quad s \in \mathbb{R}, \tag{1.19}$$

with $\kappa \in L^1(\mathbb{R})$, $z \in L^\infty(\mathbb{R})$. Let x_A denote the approximation to x obtained when (1.19) is solved with the range of integration reduced to $[-A, A]$. Then, using the results of Section 4 it is shown in [6, 16] that, under certain conditions on z,

$$|x(s) - x_A(s)| \le C \left(\frac{1}{w(s-A)} + \frac{1}{w(s+A)} \right) \operatorname*{ess.sup}_{|s| \ge A} |z(s)x(s)|, \quad |s| \le A, \tag{1.20}$$

where C is a constant depending only on κ and z and w is a weight function satisfying (1.4) which can be specified in terms of κ. In particular [6, 16], (1.20) holds with w given by (1.10), for some $\alpha \in (0,1)$ and $a, p, q \ge 0$, provided

$$w(s) \int_{\mathbb{R}\setminus[-s,s]} |\kappa(t)| dt = O(1), \quad s \to \infty.$$

Our results have a number of other significant applications. In Section 3, as noted above, we use (1.7) to show that $\Sigma_{X_0}(K) \subset \Sigma_X(K)$ if k satisfies $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$. Our results on the equivalence of spectra between X and X_w can also be exploited to shed light on equivalence of spectra for other spaces. In particular, using the denseness of X_w in $L^1(\mathbb{R}_+)$ if $\int_0^\infty w^{-1}(s) ds < \infty$, it is possible to draw conclusions about the spectrum of K as an operator on $L^p(\mathbb{R}_+)$, for $p = 1$, and then, by interpolation, for $1 < p < \infty$. See [2] for results in this direction for the case when $\boldsymbol{(A')}$ holds with $|\kappa(s)| = O(|s|^{-q})$ as $|s| \to \infty$, for some $q > 1$.

Weighted space results are also of interest for the numerical solution of (1.1). In the case that $\lambda \notin \Sigma_{X_w}(K)$ and $y \in X_w$ it holds that $x \in X_w$, i.e. that $|x(s)|w(s)$ is bounded, where x is the solution to (1.1). If this is the case and (1.1) is solved numerically, for example by a Nyström method with step-length h, to give a numerical solution x_h, it is desirable that $||x - x_h||_w \to 0$ as $h \to 0$. In particular, in the case that $|x(s)|w(s)$ is bounded below as well as above, $||x - x_h||_w \to 0$ as $h \to 0$ implies that x_h approximates x with small *relative* error in the limit $h \to 0$. See [9] for a discussion of conditions on the kernel and Nyström scheme which ensure $x_h \in X_w$ and that $||x - x_h||_w \to 0$.

The applications we have described briefly, from [2] and [9], rely on the weighted space theory for power weights of [4, 5, 3], and so assume that $\boldsymbol{(A')}$ holds with $\kappa(s) = O(|s|^{-q})$ as $|s| \to \infty$, for some $q > 1$, and that $w(s) = (1+s)^p$, for some $p \in (0, q]$. As discussed above, in the present paper we extend previous results to general classes of kernels and weight functions. This suggests that the results of [2] and [9] should be capable of generalisation to much less stringent assumptions on the kernel k and/or the weight function w.

2. The spectrum and essential spectrum of K in weighted spaces

Let K_w denote the integral operator defined by

$$K_w = M_w K M_{w^{-1}}, \tag{2.1}$$

where, for $w \in C(\mathbb{R}_+)$, M_w is the operation of multiplication by w. K_w is an integral operator of the form (1.2) and has the kernel given by (1.16). Since $M_w : X_w \to X$ is an isometric isomorphism with inverse $M_{w^{-1}}$, it is easy to see that

$$K_w \in B(X) \Leftrightarrow K \in B(X_w), \quad \lambda - K_w \in \Phi(X) \Leftrightarrow \lambda - K \in \Phi(X_w), \tag{2.2}$$

where $\Phi(X)$ denotes the set of Fredholm operators on X. Further, if both $\lambda - K$ and $\lambda - K_w$ are Fredholm, then their indices are the same. Clearly, also

$$(\lambda - K_w)^{-1} \in B(X) \Leftrightarrow (\lambda - K)^{-1} \in B(X_w). \tag{2.3}$$

Combining (2.2) with (1.3), we obtain the following characterization of the boundedness of K on X_w.

Proposition 2.1. *Suppose that the kernel k satisfies Assumptions $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$. Then $K \in B(X_w)$ if and only if*

$$\sup_{s \in \mathbb{R}_+} \int_0^s |k_w(s,t)|\, dt = \sup_{s \in \mathbb{R}_+} \int_0^s \left| \frac{w(s)}{w(t)} k(s,t) \right| dt < \infty, \tag{2.4}$$

in which case k_w also satisfies $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$.

Proof. Since w is continuous and bounded away from zero, it is easy to see that if k satisfies $\boldsymbol{(B)}$, so must k_w. Noting that $w(s)/w(t) \le 1$ if $s \le t$, we obtain (provided k satisfies $\boldsymbol{(A)}$) that k_w satisfies $\boldsymbol{(A)}$ if and only if (2.4) holds. $\square$

In the remainder of the paper we assume that k satisfies Assumption $\boldsymbol{(A')}$ with $\kappa \in L^1(\mathbb{R})$, which implies that $\boldsymbol{(A)}$ holds. We consider the case when the following assumption holds:

$$\boldsymbol{(C)} \qquad \int_0^s \frac{|\kappa(s-t)|}{w(t)}\, dt = \int_0^s \frac{|\kappa(t)|}{w(s-t)}\, dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } s \to \infty.$$

Clearly, Proposition 2.1 has the following corollary.

Corollary 2.2. *Suppose that the kernel k satisfies $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$. Then $K \in B(X_w)$ if $\boldsymbol{(C)}$ holds. In the Wiener-Hopf case $k(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, $K \in B(X_w)$ if and only if $\boldsymbol{(C)}$ holds.*

We note some simple consequences of condition $\boldsymbol{(C)}$. Firstly, it follows from $\boldsymbol{(C)}$ that (1.11) holds and that, for every $A > 0$,

$$\frac{1}{w(s-A)} \int_A^{2A} |\kappa(t)|\, dt = O\Big(\frac{1}{w(s)}\Big), \quad s \to \infty. \tag{2.5}$$

Unless $\kappa(t) = 0$ for almost all $t > 0$, it holds that

$$\int_A^{2A} |\kappa(t)|\, dt > 0$$

for some $A > 0$, so that (2.5) implies that

$$\frac{w(s)}{w(s-A)} = O(1), \quad s \to \infty, \tag{2.6}$$

for some $A > 0$. But it is clear that (2.6) must then hold for all $A > 0$.

Let us introduce at this point two additional assumptions which play a key role in the arguments in this section:

$$\boldsymbol{(E)} \qquad \sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt = O(1), \quad \text{as } A \to \infty; \tag{2.7}$$

$$\boldsymbol{(F)} \qquad \frac{w(s+1)}{w(s)} = O(1) \quad \text{as } s \to \infty. \tag{2.8}$$

Clearly, $\boldsymbol{(F)}$ limits the growth of w, implying that

$$w(s) \le Ce^{bs}, \quad s \in \mathbb{R}_+,$$

for some constants $C > 0$ and $b > 0$.

The next lemma shows that $\boldsymbol{(A')}$, $\boldsymbol{(B)}$, $\boldsymbol{(E)}$ and $\boldsymbol{(F)}$ are sufficient conditions to ensure that $K \in B(X_w)$. We will see shortly that, if $\boldsymbol{(E)}$ and $\boldsymbol{(F)}$ are replaced by slightly stronger conditions ($\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ below), then also (1.7) and (1.8) hold.

Lemma 2.3. *Assumption* $(\boldsymbol{C})$ *implies* $(\boldsymbol{E})$*. Unless* $\kappa(s)=0$ *for almost all* $s>0$*,* $(\boldsymbol{C})$ *also implies* $(\boldsymbol{F})$*. Conversely,* $(\boldsymbol{E})$ *and* $(\boldsymbol{F})$ *together imply* $(\boldsymbol{C})$*. Thus, if* k *satisfies* $(\boldsymbol{A'})$*, with* $\kappa \in L^1(\mathbb{R})$*, and* $(\boldsymbol{B})$*,* $(\boldsymbol{E})$ *and* $(\boldsymbol{F})$ *hold, then* $K \in B(X_w)$*. In the Wiener-Hopf case* $k(s,t)=\kappa(s-t)$*, with* $\kappa \in L^1(\mathbb{R})$*, it holds that* $K \in B(X_w)$ *if and only if* $(\boldsymbol{E})$ *and* $(\boldsymbol{F})$ *are satisfied or* $\kappa(s)=0$ *for almost all* $s>0$*.*

Proof. The first two assertions are immediate from the definitions and the discussion in the preceding paragraph. We thus start by proving that $(\boldsymbol{E})$ and $(\boldsymbol{F})$ imply $(\boldsymbol{C})$. Note that $(\boldsymbol{E})$ implies that, for some $A>0$ and $C>0$,

$$\int_A^{s-A} \frac{|\kappa(s-t)|}{w(t)}\,dt \le \frac{C}{w(s)}, \quad s \ge 2A. \tag{2.9}$$

From this inequality it follows that

$$\frac{1}{w(2A)} \int_A^{2A} |\kappa(s-t)|\,dt \le \int_A^{2A} \frac{|\kappa(s-t)|}{w(t)}\,dt \le \frac{C}{w(s)}, \quad s \ge 3A.$$

Thus, for $s \ge 2A$,

$$\begin{aligned} \int_0^A \frac{|\kappa(s-t)|}{w(t)}\,dt &\le \int_0^A |\kappa(s-t)|\,dt = \int_A^{2A} |\kappa(s+A-t)|\,dt \\ &\le C\frac{w(2A)}{w(s+A)} \le C\frac{w(2A)}{w(s)}. \end{aligned} \tag{2.10}$$

Also, by Assumption $(\boldsymbol{F})$, for some $C>0$, $w(s)/w(s-A) \le C$, $s \ge A$, so that

$$\int_{s-A}^{s} \frac{|\kappa(s-t)|}{w(t)}\,dt \le \frac{1}{w(s-A)} \int_0^A |\kappa(t)|\,dt \le \frac{C\|\kappa\|_1}{w(s)}. \tag{2.11}$$

Combining inequalities (2.9) through (2.11) we see that $(\boldsymbol{E})$ and $(\boldsymbol{F})$ imply $(\boldsymbol{C})$. The rest of the lemma follows from Corollary 2.2. □

We now turn our attention to the Fredholm and invertibility properties of $\lambda - K$ on X_w. Because of equations (2.2) and (2.3) we are able to relate the invertibility and Fredholm properties of $\lambda - K$ on X to those of $\lambda - K$ on X_w by comparing the operators $\lambda - K$ and $\lambda - K_w$ acting on X. The difference between these two operators is $K - K_w$, an integral operator of the form (1.2) with kernel $k - k_w$. In many cases, for example [5] if $\kappa(s) = O(s^{-q})$ as $s \to \infty$ for some $q > 1$ and $w(s) = (1+s)^p$, with $0 < p < q$, it holds that $K - K_w$ is compact on X, so that $\lambda - K$ is Fredholm if and only if $\lambda - K_w$ is Fredholm. To obtain the sharpest results, i.e. to show (1.7) and (1.8) for the widest class of weight functions w, it will prove important also to consider cases when $K - K_w$ is not compact.

For this purpose we remark that it is shown in [4] that if the integral operator K is a compact operator on X then, necessarily, its kernel k satisfies $(\boldsymbol{A})$, $(\boldsymbol{B})$ and the following additional requirement:

$$(\boldsymbol{D}) \qquad \sup_{s\ge 0} \int_A^\infty |k(s,t)|\,dt \to 0, \quad \text{as } A \to \infty.$$

The converse is not true. In particular, an example in [4] shows that if $w(s) = (1+s)^q$ and $q > 1$, $k - k_w$ can satisfy $(\boldsymbol{A})$, $(\boldsymbol{B})$ and $(\boldsymbol{D})$ with $K - K_w$ not compact. But recently [3] the following perturbation theorem has been established, showing that it is almost as useful to show that $k - k_w$ satisfies $(\boldsymbol{D})$ as to show that $K - K_w$ is compact.

Theorem 2.4. *Suppose K, L are two integral operators of the form* (1.2) *with kernels k, l satisfying conditions $(\boldsymbol{A})$, $(\boldsymbol{B})$ and l also satisfying $(\boldsymbol{D})$. Then LK is a compact operator on X. If, in addition, $\lambda \neq 0$, then $\lambda + L \in \Phi(X)$ with index zero,*

$$\lambda - K + L \in \Phi(X) \Leftrightarrow \lambda - K \in \Phi(X), \tag{2.12}$$

and if the operators in (2.12) are both Fredholm then their indices are the same.

Clearly, we set $L := K - K_w$ and hope to find conditions on k so that $k - k_w$ satisfies $(\boldsymbol{A})$, $(\boldsymbol{B})$ and $(\boldsymbol{D})$. Let us consider first the Wiener-Hopf case when $k(s,t) = \kappa(s-t)$ for some $\kappa \in L^1(\mathbb{R})$. Since

$$\frac{w(s)}{w(t)} = \left|1 - \frac{w(s)}{w(t)}\right| + 1, \quad 0 \le t \le s,$$

we have that

$$\begin{aligned} \sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \quad &\le \quad \sup_{s \ge A} \int_A^{s-A} \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt \\ &\qquad + \sup_{s \ge 2A} \int_A^{s-A} |\kappa(s-t)|\, dt. \end{aligned}$$

Now, for $s \ge 2A$,

$$\int_A^{s-A} |\kappa(s-t)|\, dt \le \int_A^{\infty} |\kappa(u)|\, du \to 0$$

as $A \to \infty$. Thus, in the Wiener-Hopf case $k(s,t) = \kappa(s-t)$, if $k - k_w$ satisfies $(\boldsymbol{D})$ then the following stronger version of $(\boldsymbol{E})$ holds:

$$(\boldsymbol{E'}) \qquad \sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \to 0, \quad \text{as } A \to \infty.$$

In the Wiener-Hopf case, if κ does not vanish a.e., and $k - k_w$ satisfies $(\boldsymbol{D})$, then a stronger version of $(\boldsymbol{F})$ also holds, namely

$$(\boldsymbol{F'}) \qquad \frac{w(s+1)}{w(s)} \to 1, \quad \text{as } s \to \infty.$$

This assumption, stronger than $(\boldsymbol{F})$, limits the growth of w still further, implying that for every $b > 0$,

$$w(s) = o(e^{bs}), \quad s \to \infty.$$

To see that $(\boldsymbol{D})$ implies $(\boldsymbol{F'})$ in the Wiener-Hopf case, suppose that $(\boldsymbol{F'})$ does not hold. Then since, for all $\delta > 0$, $w(s+1)/w(s) \to 1$ as $s \to \infty$ if and only if

$w(s+\delta)/w(s) \to 1$ as $s \to \infty$, it follows that for every $\delta > 0$ there exists $\epsilon > 0$ and a sequence (s_n) of positive numbers with $s_n \to \infty$ as $n \to \infty$ such that

$$\frac{w(s_n+\delta)}{w(s_n)} \geq 1+\epsilon, \quad n \in \mathbb{N}.$$

It follows that, for every n,

$$\frac{w(s_n+\delta)}{w(t)} \geq 1+\epsilon, \quad 0 \leq t \leq s_n, \qquad \frac{w(s_n)}{w(t)} \leq \frac{1}{1+\epsilon}, \quad t \geq s_n+\delta.$$

Now, if $k(s,t) = \kappa(s-t)$ and $k - k_w$ satisfies $\boldsymbol{(D)}$ then for every $\eta > 0$ there exists $A > 0$ such that

$$\sup_{s \geq 0} \int_A^\infty \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt < \eta.$$

This implies that, for every n for which $s_n > A$, we have

$$\epsilon \int_A^{s_n} |\kappa(s_n+\delta-t)|\, dt < \eta, \quad \frac{\epsilon}{1+\epsilon} \int_{s_n+\delta}^\infty |\kappa(s_n-t)|\, dt < \eta.$$

Since $s_n \to \infty$ as $n \to \infty$, it follows that

$$\left(\int_{-\infty}^{\delta} + \int_\delta^\infty\right) |\kappa(t)|\, dt < \eta \frac{1+\epsilon}{\epsilon},$$

for all $\eta > 0$. Thus $\kappa(t) = 0$ for almost all t with $|t| > \delta$ and, since this holds for every $\delta > 0$, we have that $\kappa = 0$.

In the proof of the following theorem, we show that, conversely, $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ are sufficient conditions to ensure that $k - k_w$ satisfies $\boldsymbol{(D)}$ whenever $\boldsymbol{(A')}$ holds.

Theorem 2.5. *Suppose k and w satisfy Assumptions $\boldsymbol{(A')}$, $\boldsymbol{(B)}$, $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$, with $\kappa \in L^1(\mathbb{R})$ in $\boldsymbol{(A')}$. Then the difference kernel $k - k_w$ satisfies conditions $\boldsymbol{(A)}$, $\boldsymbol{(B)}$ and $\boldsymbol{(D)}$. In the Wiener-Hopf case $k(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, $k - k_w$ satisfies $\boldsymbol{(A)}$, $\boldsymbol{(B)}$ and $\boldsymbol{(D)}$ if and only if κ and w satisfy $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ or $\kappa = 0$.*

Proof. If k and w satisfy $\boldsymbol{(A')}$, $\boldsymbol{(B)}$, $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$, then from Lemma 2.3 and Corollary 2.2 we have that k_w satisfies $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$, so $k - k_w$ must also satisfy $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$. It remains to check whether $k - k_w$ fulfills $\boldsymbol{(D)}$.

Let $s \geq 0$ and $0 < A^* < A/2$. We have

$$\begin{aligned}
\int_A^\infty \left|\left(1 - \frac{w(s)}{w(t)}\right) k(s,t)\right| dt &\leq \int_A^\infty \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt \\
&\leq \left(\int_A^{\max\{s-A^*,A\}} + \int_{\max\{s-A^*,A\}}^{\max\{A,s+A^*\}} + \int_{s+A^*}^\infty\right) \left|1 - \frac{w(s)}{w(t)}\right| |\kappa(s-t)|\, dt. \quad (2.13)
\end{aligned}$$

We use $(\boldsymbol{E'})$ to bound the first integral on the right hand side of equation (2.13). Note that it is non-zero only if $s \geq A+A^*$. Further, if $s \geq A+A^* > 2A^*$ then

$$\int_A^{\max\{s-A^*,A\}} \left|1-\frac{w(s)}{w(t)}\right| |\kappa(s-t)|\,dt \quad \leq \quad \int_A^{s-A^*} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \leq E_{A^*}, \quad (2.14)$$

where

$$E_{A^*} := \sup_{s\geq 2A^*} \int_{A^*}^{s-A^*} \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \to 0$$

as $A^* \to \infty$ by Assumption $(\boldsymbol{E'})$.

The second integral in (2.13) vanishes for $s \leq A - A^* < A/2$. So

$$\int_{\max\{s-A^*,A\}}^{\max\{A,s+A^*\}} \left|1-\frac{w(s)}{w(t)}\right| |\kappa(s-t)|\,dt \leq c_{A^*}(A/2)\|\kappa\|_1,$$

where

$$c_{A^*}(u) := \sup_{s\geq u} \max\left\{1-\frac{w(s)}{w(s+A^*)}, \frac{w(s)}{w(s-A^*)}-1\right\} \to 0, \quad \text{as } u \to \infty,$$

by Assumption $(\boldsymbol{F'})$.

Lastly, since $0 \leq 1 - w(s)/w(t) \leq 1$ for $t \leq s$, we have for the third integral in (2.13) that

$$\int_{s+A^*}^{\infty} \left|1-\frac{w(s)}{w(t)}\right| |\kappa(s-t)|\,dt \leq \int_{-\infty}^{-A^*} |\kappa(u)|\,du \to 0,$$

as $A^* \to \infty$.

Thus

$$\sup_{s\geq 0} \int_A^{\infty} \left|\left(1-\frac{w(s)}{w(t)}\right) k(s,t)\right| \leq E_{A^*} + \int_{-\infty}^{-A^*} |\kappa(u)|\,du + c_{A^*}(A/2)\|\kappa\|_1,$$

and, given any $\epsilon > 0$, we can choose A^* such that the sum of the first two terms on the right hand side of this inequality is less than ϵ, and then $c_{A^*}(A/2)\|\kappa\|_1 < \epsilon$ for all sufficiently large A. Thus $k - k_w$ satisfies $(\boldsymbol{D})$.

The results for the Wiener-Hopf case follow from the paragraphs preceding Theorem 2.5 or as a special case of the general result, since $(\boldsymbol{A'})$ and $(\boldsymbol{B})$ are automatically satisfied. $\square$

If the conditions of Theorem 2.5 hold we may invoke Theorem 2.4 with $L := K - K_w$ to obtain the following central theorem of the present paper. Its proof is very similar to that of Theorems 2.10 and 2.12 in [3], but for completeness we include this central point of our discussion. The argument depends on the following corollary to Theorem 2.11 in [3], bearing in mind that, as discussed in [3], Assumptions $(\boldsymbol{A})$ and $(\boldsymbol{B})$ on k ensure the assumptions of that theorem are satisfied.

Corollary 2.6. *If k satisfies $(\boldsymbol{A})$ and $(\boldsymbol{B})$, $\lambda \neq 0$, and $(\lambda - K)(X)$ is closed in X and contains all compactly supported continuous functions, then $(\lambda - K)(X) = X$.*

Theorem 2.7. *Suppose that k and w satisfy $\boldsymbol{(A')}$, $\boldsymbol{(B)}$, $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$, with $\kappa \in L^1(\mathbb{R})$ in $\boldsymbol{(A')}$. Then, for any $\lambda \in \mathbb{C}$,*

$$(\lambda - K) \in \Phi(X) \Leftrightarrow (\lambda - K_w) \in \Phi(X) \Leftrightarrow (\lambda - K) \in \Phi(X_w), \tag{2.15}$$

and if these operators are Fredholm, their indices coincide. Thus

$$\Sigma^e_X(K) = \Sigma^e_X(K_w) = \Sigma^e_{X_w}(K) \tag{2.16}$$

and it holds, moreover, that

$$\Sigma_X(K) = \Sigma_X(K_w) = \Sigma_{X_w}(K). \tag{2.17}$$

Proof. By equation (2.2), we only need to show the first equalities in equations (2.16) and (2.17). Let us first deal with the case $\lambda = 0$. Lemma 2.5 in [3] shows that K cannot be Fredholm if k satisfies $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$, so that $0 \in \Sigma^e_X(K) \subset \Sigma_X(K)$ and $0 \in \Sigma^e_{X_w}(K) \subset \Sigma_{X_w}(K)$.

We now turn our attention to the equivalence (2.15) in the case $\lambda \neq 0$. But this and the statement about the indices immediately follow from Theorem 2.4, applied with $L = K - K_w$, combined with Theorem 2.5.

To establish (2.17) note that, by what we have just shown, $(\lambda - K)^{-1} \in B(X)$ implies that $\lambda - K$ is injective and Fredholm of index zero on $X_w \subset X$. But this means that $(\lambda - K) : X_w \to X_w$ is also surjective, and thus $(\lambda - K)^{-1} \in B(X_w)$. Thus $\Sigma_{X_w}(K) \subset \Sigma_X(K)$.

For the other direction, if $(\lambda - K)^{-1} \in B(X_w)$, then $X_w \subset (\lambda - K)(X)$ and also, by (2.15), $\lambda - K$ is Fredholm of index zero on X so that $(\lambda - K)(X)$ is closed in X. From Corollary 2.6 it follows that $\lambda - K : X \to X$ is surjective. Since it has index zero, it must also be injective and thus $\lambda \notin \Sigma_X(K)$. □

As an immediate consequence of Theorem 2.7 we have the following corollary on the solvability of the integral equation (1.1).

Corollary 2.8. *Suppose k satisfies Assumptions $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$, with $\kappa \in L^1(\mathbb{R})$ in $\boldsymbol{(A')}$. Let $\mathcal{W}$ be the collection of all $w \in C(\mathbb{R}_+)$ fulfilling* (1.4) *and for which Assumptions $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ are satisfied. Assume further that, for some $w \in \mathcal{W}$, the integral equation* (1.1) *has a unique solution $x \in X_w$ for every $y \in X_w$. Then, for* all *$w \in \mathcal{W}$, equation* (1.1) *has a unique solution $x \in X_w$ for every $y \in X_w$, and*

$$\sup_{s\in\mathbb{R}_+} |w(s)x(s)| = \|x\|_w \leq C\|y\|_w = C \sup_{s\in\mathbb{R}_+} |w(s)y(s)|,$$

where C is a positive constant depending only on w, k and λ.

In the special case that $k(s,t) = 0$ for $0 \leq t \leq s$, by Proposition 2.1, $K \in B(X_w)$ for every w satisfying (1.4), as we have observed already for the Wiener-Hopf case in Lemma 2.3. Slightly more can be said about the relationship between $\Sigma_X(K)$ and $\Sigma_{X_w}(K)$ in this case.

Theorem 2.9. *If k satisfies $\boldsymbol{(A)}$ and $\boldsymbol{(B)}$ and $k(s,t) = 0$ for $0 \leq t \leq s$, then $K \in B(X_w)$ and*

$$\Sigma_{X_w}(K) \subset \Sigma_X(K). \tag{2.18}$$

If also $\boldsymbol{(F')}$ *holds and* k *satisfies* $\boldsymbol{(A')}$ *for some* $\kappa \in L^1(\mathbb{R})$*, in which case* κ *can be chosen with* $\kappa(s) = 0, s > 0$*, then Assumption* $\boldsymbol{(E')}$ *holds so that Theorem 2.7 applies and, in particular,* (2.16) *and* (2.17) *hold.*

Remark 2.10. This result shows that, if k satisfies $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$, with $\kappa(s) = 0$, $s > 0$, then (2.18) holds, and that if also w satisfies $\boldsymbol{(F')}$ then (2.16) and (2.17) hold. Example 2.11 below shows that, if w satisfies $\boldsymbol{(F)}$ but not $\boldsymbol{(F')}$, then no stronger relationship between spectra than (2.18) need hold. In particular, it need not hold that $\Sigma_{X_w}(K) = \Sigma_X(K)$ nor that $\Sigma^e_{X_w}(K) \subset \Sigma^e_X(K)$.

Proof. Let $u > 0$ and define for every $y \in X$ the function $y_u \in X$ by setting $y_u(s) = y(s)$ for $s \geq u$ and $y_u(s) = y(u)$ for all $0 \leq s < u$. Then $\|y_u\| = \sup_{s\geq u} |y(s)|$.

Suppose k satisfies the assumptions of the theorem. Then $K \in B(X)$ so that for any $x \in X$ we have

$$|Kx(s)| = |Kx_s(s)| \leq \|K\|\|x_s\| \leq \|K\| \sup_{t\geq s} |x(t)|, \quad s \in \mathbb{R}_+.$$

Hence $K \in B(X_w)$ with norm not larger than $\|K\|$.

To prove (2.18) let us assume that $\lambda \notin \Sigma_X(K)$, i.e. $(\lambda - K)^{-1} \in B(X)$. Then, for every $y \in X$ the integral equation

$$\lambda x(s) - \int_s^\infty k(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R}_+. \tag{2.19}$$

has a unique solution $x \in X$ and $\|x\| \leq C\|y\|$.

Let $u > 0$ and $y \in X$. Denote by x, x^u the unique solution of (2.19) with right-hand side y, y_u, respectively. We shall see in a moment that

$$x(s) = x^u(s), \quad s \geq u, \tag{2.20}$$

holds, so that

$$\sup_{s\geq u} |x(s)| \leq \|x^u\| \leq C\|y_u\| = C \sup_{s\geq u} |y(u)|.$$

Thus, if $y \in X_w$ then $x \in X_w$ with $\|x\|_w \leq C\|y\|_w$. Hence $(\lambda - K)^{-1} \in B(X_w)$, i.e. $\lambda \notin \Sigma_{X_w}(X)$ which is what we set out to show.

It remains to prove that (2.20) is true. To this end let us show that the integral equation

$$\lambda\tilde{x}(s) - \int_{\max\{s,u\}}^\infty k(s,t)\tilde{x}(t)\,dt = \tilde{y}(s), \quad s \in \mathbb{R}_+. \tag{2.21}$$

has a unique solution $\tilde{x} \in X$ for every $\tilde{y} \in X$. Denote the kernel of the integral operator K_+ in (2.21) by k_+, so that

$$k_+(s,t) = \begin{cases} 0, & 0 \leq t < u, \\ k(s,t), & u \leq t, \end{cases} \quad s \in \mathbb{R}_+.$$

Also, set $k_- := k - k_+$. It is not hard to see that k_- satisfies Assumptions $\boldsymbol{(A)}$, $\boldsymbol{(B)}$ and $\boldsymbol{(D)}$. We apply Theorem 2.4 with $L = K - K_+$ to see that $\lambda - K_+$ is Fredholm of index 0 since $\lambda - K$ (as an invertible operator) is Fredholm of index 0.

To see that $\lambda - K_+$ is also surjective, choose any $\tilde{y} \in X$ and let $x := (\lambda - K)^{-1}\tilde{y}$ and set

$$\tilde{x}(s) := \begin{cases} x(s) - \frac{1}{\lambda}\int_s^u k(s,t)x(t)\,dt, & 0 \le s < u, \\ x(s), & s \ge u. \end{cases}$$

Then $\tilde{x} \in X$ and $(\lambda - K_+)\tilde{x} = (\lambda - K)x = \tilde{y}$ and thus $\lambda - K_+$ is surjective, whence $(\lambda - K_+)^{-1} \in B(X)$.

For the last step, we define the function z by

$$z(s) := \frac{1}{\lambda}\int_{\max\{s,u\}}^{\infty} k(s,t)\big(x(t) - x^u(t)\big)\,dt, \quad s \in \mathbb{R}_+.$$

Then, by the definition of x and x^u,

$$z(s) = x(s) - x^u(s), \quad s \ge u. \tag{2.22}$$

Thus $\lambda z = K_+ z$ and, since $(\lambda - K_+)$ is injective, $z = 0$; now (2.22) implies that (2.20) must indeed be true and the theorem follows. □

We now comment further on the necessity of the requirement $(\boldsymbol{F'})$ in the Wiener-Hopf case $k(s,t) = \kappa(s-t)$. We have seen in Lemma 2.3 that, unless κ vanishes on the positive half-line, necessarily $(\boldsymbol{F})$ holds in this case if $K \in B(X_w)$. We have seen also that our method of argument, based on Theorem 2.4 applied with $L = K - K_w$, so that $k - k_w$ must satisfy $(\boldsymbol{D})$, requires that w satisfies the stronger condition $(\boldsymbol{F'})$. Thus $(\boldsymbol{F'})$ is a necessary condition for $K - K_w$ to be compact, though not, as discussed above, a sufficient condition. But the question arises as to whether, in the Wiener-Hopf case, Assumption $(\boldsymbol{F'})$ is also necessary for the results of Theorem 2.7 to hold.

We can give a partial answer to this question by considering the weight function $w(s) = \exp(bs)$, $b > 0$, which satisfies $(\boldsymbol{F})$ but not $(\boldsymbol{F'})$. In this case, if $k(s,t) = \kappa(s-t)$ with $\kappa \in L^1(\mathbb{R})$, then $k_w(s,t) = \kappa_b(s-t)$ with $\kappa_b(s) := \kappa(s)\exp(bs)$. Thus

$$K \in B(X_w) \iff \int_0^\infty |\kappa(t)|e^{bt}\,dt < \infty. \tag{2.23}$$

Further, if (2.23) holds, then, from (1.13) and (1.14) applied with $\kappa = \kappa_b$, we deduce that

$$\Sigma^e_{X_w}(K) = \{\hat{\kappa}(\xi - ib) : \xi \in \mathbb{R}\} \cup \{0\} \tag{2.24}$$

and

$$\Sigma_{X_w}(K) = \Sigma^e_{X_w}(K) \cup \{\lambda : [\arg(\lambda - \hat{\kappa}(\xi - ib))]_{-\infty}^{\infty} \neq 0\}, \tag{2.25}$$

with $\hat{\kappa}$ defined by (1.15). Thus we have explicit expressions in this case for the spectrum and essential spectrum of K as an operator on both X and X_w and can check for a particular choice of κ whether these spectra coincide, i.e. whether (2.16) and (2.17) hold. We point out that, if (2.23) holds, then

$$\sup_{s \ge 2A} \int_A^{s-A} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt = \int_A^\infty e^{bt}|\kappa(t)|\,dt \to 0$$

as $A \to \infty$, so that $\boldsymbol{(E')}$ holds. Thus all the conditions of Theorem 2.7 are satisfied in this case, except that $\boldsymbol{(F')}$ is replaced by the weaker $\boldsymbol{(F)}$.

The following examples illustrate the range of possible behaviour. The first example shows that there exists a large class of κ for which (2.16) and (2.17) do hold, while the second example shows that (2.16) and (2.17) do not hold for a large class of κ. The third example is a case in which $\kappa(s) = 0$, $s > 0$, and (2.16) and (2.17) do not hold, although, by Theorem 2.9, (2.18) applies.

Example 2.11. *Suppose that f is real and even and that*

$$\int_0^\infty e^{bs/2}\big(|f(s)| + |f(-s)|\big)\, ds < \infty.$$

Then $\hat{f}(\xi)$ is analytic in the strip $|\mathrm{Im}\,\xi| < b/2$ and continuous in $|\mathrm{Im}\,\xi| \le b/2$. Further $\hat{f}(\xi) = \hat{f}(-\xi)$, $|\mathrm{Im}\,\xi| \le b/2$. Define $\kappa(s) := e^{-bs/2} f(s)$. Then (2.23) *holds and*

$$\hat{\kappa}(\xi) = \hat{f}(\xi + ib/2), \quad \hat{\kappa}(\xi - ib) = \hat{f}(\xi - ib/2), \quad \xi \in \mathbb{R}.$$

Thus, and from (1.13), (1.14), (2.24) *and* (2.25) *it follows that* (2.16) *and* (2.17) *in Theorem 2.7 hold.*

If $\lambda - K$ is Fredholm on X then its index (see e.g. [10, 15]*) is $\gamma := \frac{1}{2\pi}[\arg(\lambda - \hat{\kappa}(\xi))]_{-\infty}^{\infty}$ so that the index of $\lambda - K$ on X_w is $\frac{1}{2\pi}[\arg(\lambda - \hat{\kappa}(\xi - ib))]_{-\infty}^{\infty} = \frac{1}{2\pi}[\arg(\lambda - \hat{\kappa}(-\xi))]_{-\infty}^{\infty} = -\gamma$. Thus the other conclusion of Theorem 2.7 does not hold in this case since, if $\lambda - K$ is Fredholm on X and X_w, its index on X is the negative of its index on X_w.*

Example 2.12. *Suppose that κ is real and even and that* (2.23) *holds. Then $\hat{\kappa}(\xi)$ is real and even so that*

$$\Sigma_X(K) = \Sigma_X^e(K) = [\kappa_-, \kappa_+],$$

where $\kappa_- = \inf_{\xi \in \mathbb{R}} \hat{\kappa}(\xi)$, $\kappa_+ = \sup_{\xi \in \mathbb{R}} \hat{\kappa}(\xi)$. But

$$\hat{\kappa}(\xi - ib) = \int_{-\infty}^\infty \kappa(s) e^{bs} \cos(\xi s)\, ds + 2i \int_0^\infty \kappa(s) \sinh(bs) \sin(\xi s)\, ds, \quad \xi \in \mathbb{R}.$$

The imaginary part of $\hat{\kappa}(\xi - ib)$ is the sine transform of $2\kappa(s)\sinh(bs)$. By the injectivity of the sine transform, unless $\kappa = 0$, $\mathrm{Im}\,\hat{\kappa}(\xi - ib) \ne 0$ for at least one $\xi \in \mathbb{R}$, so that

$$\Sigma_X^e(K) \ne \Sigma_{X_w}^e(K), \quad \Sigma_X(K) \ne \Sigma_X(K).$$

Example 2.13. *Define κ by*

$$\kappa(s) = \begin{cases} 0, & s \ge 0, \\ e^s, & s < 0. \end{cases}$$

Then (2.23) *holds for all $b > 0$ so that $K \in B(X_w)$. Also*

$$\hat{\kappa}(\xi) = \frac{1}{1 + i\xi}, \quad \hat{\kappa}(\xi - ib) = \frac{1}{1 + b + i\xi}, \quad \xi \in \mathbb{R},$$

so that

$$\Sigma_X^e(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}, \quad \Sigma_X(K) = \left\{\lambda : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

and

$$\begin{aligned} \Sigma_{X_w}^e(K) &= \left\{\lambda : \left|\lambda - \frac{1}{2(1+b)}\right| = \frac{1}{2(1+b)}\right\}, \\ \Sigma_{X_w}(K) &= \left\{\lambda : \left|\lambda - \frac{1}{2(1+b)}\right| \leq \frac{1}{2(1+b)}\right\}. \end{aligned}$$

Thus $\Sigma_X(K) \neq \Sigma_{X_w}(K)$ *and* $\Sigma_X^e(K) \neq \Sigma_{X_w}^e(K)$*; in fact* $\Sigma_X^e(K) \cap \Sigma_{X_w}^e(K) = \{0\}$*. But note that* $\Sigma_{X_w}(K) \subset \Sigma_X(K)$*, in agreement with Theorem 2.9.*

3. Sufficient conditions on kernels and examples

While in applications Assumption $(\boldsymbol{F'})$ is often easily verified, Assumption $(\boldsymbol{E'})$ is typically much harder to check. In this section we derive simpler conditions which imply that $(\boldsymbol{E'})$ holds, and give examples of kernels and weights which satisfy $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$.

In most cases of practical interest it holds that $w(s)$ is continuously differentiable, at least for all sufficiently large s, say $s \geq s_0$. In this case we have that

$$\frac{w(s)}{w(t)} = \exp\left(\int_t^s \frac{w'(u)}{w(u)}\, du\right), \quad s_0 \leq t \leq s, \tag{3.1}$$

so that, if

$$\frac{w'(s)}{w(s)} \to 0, \quad \text{as } s \to \infty,$$

then $(\boldsymbol{F'})$ holds. Of course, not every w satisfying (1.4) is differentiable, even in a weak sense. But for every $w \in C(\mathbb{R}_+)$ satisfying (1.4) the function

$$\tilde{w}(s) := \frac{\int_s^{s+1} w(t)\, dt}{\int_0^1 w(t)\, dt}, \quad s \in \mathbb{R}_+,$$

satisfies (1.4) and is continuously differentiable. Further, we have the following result.

Lemma 3.1. *Assumption* $(\boldsymbol{F})$ *holds if and only if*

$$\frac{\tilde{w}'(s)}{\tilde{w}(s)} = O(1), \quad \textit{as } s \to \infty. \tag{3.2}$$

Assumption $(\boldsymbol{F'})$ *holds if and only if*

$$\frac{\tilde{w}'(s)}{\tilde{w}(s)} \to 0, \quad \textit{as } s \to \infty. \tag{3.3}$$

If w satisfies $\boldsymbol{(F)}$ then, for some $C > 0$,

$$\frac{w(s)}{w(1)} \le \tilde{w}(s) \le Cw(s), \quad s \ge 0. \tag{3.4}$$

Proof. For $s \ge 0$,

$$\frac{w(s)}{w(1)} \le \frac{w(s)}{\int_0^1 w(t)\,dt} \le \tilde{w}(s) \le \frac{w(s+1)}{\int_0^1 w(t)\,dt} \le w(s+1),$$

so that

$$\frac{\tilde{w}'(s)}{\tilde{w}(s)} = \frac{w(s+1) - w(s)}{\tilde{w}(s)\int_0^1 w(t)\,dt} \le \frac{w(s+1)}{w(s)} - 1$$

and, for $s \ge 1$,

$$\frac{w(s+1)}{w(s)} \le w(1)\frac{\tilde{w}(s+1)}{\tilde{w}(s-1)} = w(1)\exp\left(\int_{s-1}^{s+1} \frac{\tilde{w}'(t)}{\tilde{w}(t)}\,dt\right).$$

From these inequalities the equivalence of $\boldsymbol{(F)}$ and (3.2) and also that of $\boldsymbol{(F')}$ and (3.3) follows. Further, if $\boldsymbol{(F)}$ holds then, for some $C > 0$, $w(s+1) \le Cw(s)$, $s \ge 0$, so that (3.4) is true. $\square$

In view of this result, in order to check that $\boldsymbol{(E)}$ and $\boldsymbol{(F)}$ hold, or that $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ hold, it is sufficient to check that $\tilde{w}$ satisfies (3.2) or (3.3), respectively, and that $\boldsymbol{(E)}$ or $\boldsymbol{(E')}$, respectively, hold with w replaced by $\tilde{w}$. We will assume in the remainder of this section, when deriving conditions which ensure that $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ hold, that $w(s)$ is continuously differentiable for all sufficiently large s. The reader should bear in mind that if $\tilde{w}$, which is necessarily continuously differentiable, satisfies the conditions we require in the various propositions below, then $\tilde{w}$ satisfies (3.3), (3.4) and $\boldsymbol{(E')}$ and hence, by Lemma 3.1, w satisfies $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$.

Our first two propositions deal with the case when $w'(s)/w(s)$ is bounded by θ/s for some $\theta > 0$ and all sufficiently large s. Note that we have then the bound

$$1 \le \frac{w(s)}{w(t)} \le \exp\left(\int_t^s \frac{\theta}{u}\,du\right) = \left(\frac{s}{t}\right)^\theta \tag{3.5}$$

if $s \ge t$ and t is sufficiently large. Keeping t fixed in this equation, we see that in this case necessarily $w(s) = O(s^\theta)$, $s \to \infty$.

Proposition 3.2. *Suppose that k satisfies $\boldsymbol{(A')}$, with $\kappa \in L^1(\mathbb{R})$, and that there exists $\theta > 0$ such that for all sufficiently large s the inequality*

$$\frac{w'(s)}{w(s)} \le \frac{\theta}{s} \tag{3.6}$$

holds. Further, suppose that either

$$w^{-1} \in L^1(\mathbb{R}_+) \quad \text{and} \quad \lambda(s) := \int_s^{s+1} |\kappa(t)|\,dt = O\Big(\frac{1}{w(s)}\Big), \quad s \to \infty, \tag{3.7}$$

or, alternatively,

$$w(s) \int_s^\infty |\kappa(t)|\, dt = O(1), \quad s \to \infty, \tag{3.8}$$

holds. Then Assumptions **(E′)** *and* **(F′)** *are satisfied.*

Proof. That **(F′)** holds follows from (3.1). Note that, for $1 \le u \le s$,

$$\int_{u-1}^{u} \frac{|\kappa(t)|}{w(s-t)}\, dt \le \frac{\lambda(u-1)}{w(s-u)} \le \lambda(u-1) \int_u^{u+1} \frac{dt}{w(s-t)}. \tag{3.9}$$

Thus, if (3.7) holds, then, for some $C > 0$, $w(s)\lambda(s) \le C$ for $s \ge 0$, and we obtain, for A sufficiently large and $s \ge 2A$, the bound

$$\begin{aligned} \int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt &= w(s) \int_{s/2}^{s-A} \frac{|\kappa(t)|}{w(s-t)}\, dt \\ &\le w(s) \sup_{t \ge s/2-1} \lambda(t) \int_{s/2}^{s-A+1} \frac{dt}{w(s-t)} \le C \frac{w(s)}{w(s/2-1)} \int_{A-1}^\infty \frac{dt}{w(t)}. \end{aligned} \tag{3.10}$$

Note that, by our assumption (3.6), the inequality (3.5) holds for $s \ge t$ and t large enough. Hence, and from (3.10), for all sufficiently large A,

$$\sup_{s \ge 2A} \int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \le C \left(\frac{2}{1 - A^{-1}} \right)^\theta \int_{A-1}^\infty \frac{dt}{w(t)} \to 0, \tag{3.11}$$

as $A \to \infty$.

In the other case, when assumption (3.8) holds, inequality (3.5) implies that for all sufficiently large A and $s \ge 2A$

$$\begin{aligned} &\int_A^{s/2} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \le \\ &\qquad \frac{w(s)}{w(A)} \int_A^{s/2} |\kappa(s-t)|\, dt \le 2^\theta \frac{w(s/2)}{w(A)} \int_{s/2}^\infty |\kappa(t)|\, dt \to 0, \end{aligned} \tag{3.12}$$

as $A \to \infty$, uniformly in $s \ge 2A$.

Further, in both cases, for all sufficiently large A it holds that

$$\sup_{s \ge 2A} \int_{s/2}^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \le 2^\theta \int_A^\infty |\kappa(t)|\, dt \to 0,$$

as $A \to \infty$. Thus **(E′)** holds. □

If the constant θ in the bound for $w'(s)/w(s)$ is in the interval $(0, 1]$, then $1 \le w(s) = O(s)$ as $s \to \infty$, so that w^{-1} is not integrable. Thus condition (3.7) of the previous proposition is not satisfied, and Proposition 3.2 applies only if (3.8) holds. Consider now the example when $\kappa(s) = (1 + |s|)^{-3/2}$ and $w(s) = (1 + s)^{3/4}$. Then

$$w(s) \int_s^\infty |\kappa(t)|\, dt = 2(1 + s)^{1/4},$$

which is clearly unbounded as $s \to \infty$, so that neither of the two conditions on κ in Proposition 3.2 is applicable. The next proposition gives alternative conditions on κ when $\theta \le 1$ which apply to this example.

Proposition 3.3. *Suppose that k satisfies Assumption **(A′)**, with $\kappa \in L^1(\mathbb{R})$, and that, for some $\theta \in (0,1]$,*

$$\frac{w'(s)}{w(s)} \le \frac{\theta}{s}, \tag{3.13}$$

for all sufficiently large s, and

$$\lambda(s) := \int_s^{s+1} |\kappa(t)|\, dt = \begin{cases} O(s^{-1}) & , \text{ if } \theta < 1, \\ o((s \ln s)^{-1}) & , \text{ if } \theta = 1, \end{cases} \quad \text{as } s \to \infty.$$

*Then Assumptions **(E′)** and **(F′)** are satisfied.*

Proof. Since (3.13) holds for all sufficiently large s, it follows that, for some $M > 0$, (3.5) holds for $s \ge t \ge M$. Further, if $\theta < 1$, then, for some $C > 0$,

$$s\lambda(s) \le C, \quad s \in \mathbb{R}_+. \tag{3.14}$$

Suppose $A > M + 1$ and $\eta \in (0, 1/2]$. Then, for $s \ge 2A$,

$$\int_{\max\{A,\eta s\}}^{s-A} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt \le \left(\frac{s}{\eta s}\right)^{\theta} \int_{\max\{A,\eta s\}}^{s-A} |\kappa(s-t)|\, dt \le \eta^{-\theta} \int_A^{\infty} |\kappa(t)|\, dt. \tag{3.15}$$

Further, for $\eta s \ge A$, using (3.9) with $w(s) = s^\theta$ to obtain (3.17) from (3.16), we see that

$$\begin{aligned} \int_A^{\eta s} \frac{w(s)}{w(t)} |\kappa(s-t)|\, dt &\le \int_{s(1-\eta)}^{s-A} \frac{s^\theta |\kappa(t)|}{(s-t)^\theta}\, dt & (3.16) \\ &\le s^\theta \left(\sup_{t \ge s(1-\eta)-1} \lambda(t) \right) \int_{s(1-\eta)}^{s-A+1} \frac{dt}{(s-t)^\theta} & (3.17) \\ &\le \frac{Cs^\theta}{s(1-\eta)} \int_{A-1}^{\eta s} \frac{dt}{t^\theta}. & (3.18) \end{aligned}$$

In the case $\theta < 1$, since $0 < \eta \le \frac{1}{2}$ and $\eta s \ge A$, this expression is bounded above by

$$\frac{2Cs^\theta}{s-2} \int_0^{\eta s} \frac{dt}{t^\theta} = \frac{2sC}{(s-2)(1-\theta)} \eta^{1-\theta} \le \frac{2AC}{(A-1)(1-\theta)} \eta^{1-\theta} \le \frac{2(M+1)C}{M(1-\theta)} \eta^{1-\theta}. \tag{3.19}$$

Combining the inequalities (3.15) through (3.19), we see that, for some $C_1 > 0$ and all sufficiently large A,

$$\sup_{s\geq 2A}\int_A^{s-A}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \leq \eta^{-\theta}\int_A^\infty |\kappa(t)|\,dt + C_1\eta^{1-\theta}.$$

For any $\epsilon > 0$ we can choose first η small enough so that $\eta^{1-\theta}C_1 < \epsilon/2$ and then, for all sufficiently large A,

$$\sup_{s\geq 2A}\int_A^{s-A}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt < \eta^{-\theta}\int_A^\infty |\kappa(t)|\,dt + \frac{\epsilon}{2} < \epsilon.$$

so that $\boldsymbol{(E')}$ follows.

In the case $\theta = 1$, we set $\eta = 1/2$ and find from (3.17) that, for $A \geq 2$ and $s \geq 2A$,

$$\int_A^{s/2}\frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \leq s\Big(\sup_{t\geq\frac{s}{2}-1}\lambda(t)\Big)\int_{A-1}^{s/2}\frac{dt}{t} \leq s\Big(\sup_{t\geq\frac{s}{2}-1}\lambda(t)\Big)\ln\frac{s}{2} \to 0$$

as $s \to \infty$. Combining this bound with (3.15) we see that $\boldsymbol{(E')}$ holds. □

The following example considers the important special case of the power weight $w(s) = (1+s)^p$, sharpening, as discussed in the introduction, the results of [11, 15, 4].

Example 3.4. *Suppose $w(s) := (1+s)^p, p > 0$, and the kernel k satisfies Assumption $\boldsymbol{(A')}$ with $\kappa \in L^1(\mathbb{R})$. Then Assumption $\boldsymbol{(F')}$ holds,*

$$\frac{w'(s)}{w(s)} = \frac{p}{1+s}, \quad s \in \mathbb{R}_+,$$

and, by Propositions 3.3 and 3.2, Assumption $\boldsymbol{(E')}$ holds if

$$\int_s^{s+1}|\kappa(t)|\,dt = \begin{cases} O(s^{-p}), & p > 1,\\ o\big((s\ln s)^{-1}\big), & p = 1,\\ O(s^{-1}), & 0 < p < 1,\end{cases} \qquad \text{as } s \to \infty. \tag{3.20}$$

Thus, if (3.20) *is satisfied and k also satisfies $\boldsymbol{(B)}$, then, by Lemma 2.3 and Theorem 2.7, $K \in B(X_w)$ and the spectral equivalences* (2.16) *and* (2.17) *hold.*

In the Wiener-Hopf case $k(s,t) = \kappa(s-t)$, with $\kappa \in L^1(\mathbb{R})$, it follows from Example 3.4 that, if $w(s) = (1+s)^p$, for some $p > 0$, and (3.20) holds, then $K \in B(X_w)$ and (2.16) and (2.17) hold. As a consequence of Corollary 2.2 and since Assumption $\boldsymbol{(C)}$ implies (1.11), we have also that $K \in B(X_w)$ implies that (1.18) holds for $r = p$. Thus the statement

$$\int_s^{s+1}|\kappa(t)|\,dt = O(s^{-q}) \text{ as } s\to\infty \implies K \in B(X_w) \implies \int_s^{s+1}|\kappa(t)|\,dt = O(s^{-r}) \text{ as } s\to\infty \tag{3.21}$$

holds for $r = q = p$ if $p > 1$, for $r = 1$ and every $q > 1$ if $p = 1$, and for $r = p$ and $q = 1$ if $0 < p < 1$. In the case $0 < p < 1$ the implications (3.21) do not hold for any values of q and r with $r > p$ or $q < 1$ as shown by the following examples.

Example 3.5. *Suppose that $k(s,t) = \kappa(s-t)$ and that, for some $p > 0$,*

$$\kappa(t) = \begin{cases} t^{-p}, & e^n \le t < e^n + 1, \quad n \in \mathbb{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $\kappa \in L^1(\mathbb{R})$, in fact, for $s > 0$, where $\lfloor \ln s \rfloor$ denotes the largest integer $\le \ln s$,

$$\int_s^\infty |\kappa(t)|\, dt \le \int_{e^{\lfloor \ln s \rfloor}}^\infty |\kappa(t)|\, dt < \sum_{m=\lfloor \ln s \rfloor}^\infty e^{-pm} = \frac{e^{-p\lfloor \ln s \rfloor}}{1-e^{-p}} \le \frac{e^p s^{-p}}{1-e^{-p}}.$$

Thus, if $w(s) = (1+s)^p$, then (3.8) is satisfied and, by Proposition 3.2, $\boldsymbol{(E')}$ and $\boldsymbol{(F')}$ hold. It follows from Lemma 2.3 that $K \in B(X_w)$. But note that, for $s = e^n, n \in \mathbb{N}$,

$$\int_s^{s+1} |\kappa(t)|\, dt > (1+s)^{-p},$$

so that (1.18) holds only for $r \le p$.

Example 3.6. *Suppose that $k(s,t) = \kappa(s-t)$ and that, for some $q \in (0,1)$ and some positive sequences $(a_n), (b_n)$, with $0 < a_1 < a_1 + b_1 < a_2 < a_2 + b_2 < a_3 < \ldots$ it holds that*

$$\kappa(t) = \begin{cases} t^{-q}, & a_n \le t < a_n + b_n, n \in \mathbb{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

Further, suppose that $a - 1 > b \ge 0$, $a_n \sim n^a$, $b_n \sim n^b$ as $n \to \infty$, and $p \in (0,q)$. Then

$$\|\kappa\|_1 = \sum_{n=1}^\infty \int_{a_n}^{a_n+b_n} t^{-q}\, dt \le \sum_{n=1}^\infty b_n {a_n}^{-q} < \infty,$$

provided $aq - b > 1$. Moreover, where $w(s) = (1+s)^p$, it holds that

$$\begin{aligned} w(a_n + b_n) \int_0^{a_n+b_n} \frac{|\kappa(t)|}{w(a_n+b_n-t)}\, dt &\ge \frac{w(a_n+b_n)}{w(b_n)} \int_{a_n}^{a_n+b_n} |\kappa(t)|\, dt \\ &> \frac{w(a_n+b_n)}{w(b_n)} (a_n+b_n)^{-q} b_n \sim n^{ap-aq-bp+b} \end{aligned}$$

as $n \to \infty$. Now, suppose that we choose (a_n) and (b_n) so that $a > (1-p)/((1-q)p)$ (which ensures that $a(q-p)/(1-p) < aq - 1$) and so that $a(q-p)/(1-p) < b < aq - 1$. Then $aq - b > 1$, so that $\kappa \in L^1(\mathbb{R})$, and $ap - aq - bp + b > 0$, so that $\boldsymbol{(C)}$ does not hold, and so, by Corollary 2.2, $K \notin B(X_w)$. But note that (1.18) holds with $r = q$.

Having dealt with the case when $w'(s)/w(s)$ is bounded by a multiple of $1/s$, we now turn our attention to the case when $w'(s)/w(s)$ decays at a slower rate.

Proposition 3.7. *Suppose that k satisfies $(\boldsymbol{A'})$, with $\kappa \in L^1(\mathbb{R})$, that $w'(s)/w(s)$ is monotonic decreasing for all sufficiently large s and, for some $\alpha \in (0,1)$, we have that*

$$\frac{sw'(s)}{w(s)} \to \infty, \quad \frac{w'(s)}{w(s)} = O(s^{\alpha-1}),$$

as $s \to \infty$. Then w satisfies $(\boldsymbol{F'})$. If also

$$\lambda(s) := \int_s^{s+1} |\kappa(t)|\,dt = O\Big(\frac{1}{w(s)}\Big), \quad s \to \infty, \tag{3.22}$$

then Assumption $(\boldsymbol{E'})$ is fulfilled by k.

Proof. Choose $\beta > 1/(1-\alpha)$. By the assumptions of the proposition we have, for some $q > 0$ and all sufficiently large s,

$$\frac{\beta}{s} \le \frac{w'(s)}{w(s)} \le \frac{q}{s^{1-\alpha}}. \tag{3.23}$$

Thus, for $s \ge t$ and t large enough,

$$\begin{aligned}\left(\frac{s}{t}\right)^{\beta} = \exp\left(\int_t^s \frac{\beta}{u}\,du\right) \le \exp\left(\int_t^s \frac{w'(u)}{w(u)}\,du\right) = \frac{w(s)}{w(t)} \\ \le \exp\left(\int_t^s \frac{q}{u^{1-\alpha}}\,du\right) \le \exp\big(q(s-t)t^{\alpha-1}\big).\end{aligned} \tag{3.24}$$

Keeping t fixed in this equation, we see that $s^{1/(1-\alpha)}/w(s) \to 0$ as $s \to \infty$ so that $w^{-1} \in L^1(\mathbb{R}_+)$ and

$$\frac{s}{w(s^{1-\alpha})} \to 0, \quad \text{as } s \to \infty. \tag{3.25}$$

Now, for all u sufficiently large and $s \ge u$, we get from (3.9), our assumption (3.22) on κ and the fact that $w(s)/w(t)$ is bounded for $|s-t| \le 1$ when s is large enough, the bound

$$\begin{aligned}\int_{u-1}^{u} \frac{|\kappa(t)|}{w(s-t)}\,dt \le \int_u^{u+1} \frac{\lambda(u-1)}{w(s-t)}\,dt \le \\ \int_u^{u+1} \frac{C}{w(t-1)w(s-t)}\,dt \le \int_u^{u+1} \frac{C_1}{w(t)w(s-t)}\,dt,\end{aligned}$$

where C and C_1 are some positive constants. Thus, if $A > 0$ is large enough and $s \ge 2A$, we obtain

$$\begin{aligned}\int_A^{s-A} \frac{w(s)}{w(t)}|\kappa(s-t)|\,dt \le \\ C_1 w(s)\int_{A-1}^{s-A+1} \frac{dt}{w(t)w(s-t)} = 2C_1 w(s)\int_{A-1}^{s/2} \frac{dt}{w(t)w(s-t)}.\end{aligned} \tag{3.26}$$

Now, for all sufficiently large s, from (3.24),

$$\frac{w(s)}{w(s-s^{1-\alpha})} \le C_2, \tag{3.27}$$

where C_2 is some positive constant. Thus, if A is large enough and $s^{1-\alpha} \ge A-1$,

$$w(s)\int_{A-1}^{s^{1-\alpha}} \frac{dt}{w(s-t)w(t)} \le \frac{w(s)}{w(s-s^{1-\alpha})}\int_{A-1}^{\infty}\frac{dt}{w(t)} \le C_2\int_{A-1}^{\infty}\frac{dt}{w(t)}. \tag{3.28}$$

Further, by the monotonicity of $w'(s)/w(s)$ for large argument we get that

$$\frac{d}{dt}\big(w(t)w(s-t)\big) = w(t)w(s-t)\left(\frac{w'(t)}{w(t)} - \frac{w'(s-t)}{w(s-t)}\right) \ge 0, \quad s^{1-\alpha} \le t \le s/2,$$

when s is large enough. Thus, for all sufficiently large s,

$$w(s)\int_{s^{1-\alpha}}^{s/2}\frac{dt}{w(t)w(s-t)}\,dt \le \frac{s}{2}\,\frac{w(s)}{w(s^{1-\alpha})w(s-s^{1-\alpha})} \to 0, \tag{3.29}$$

as $s\to\infty$ from (3.27) and (3.25). From (3.26), (3.28) and (3.29) we conclude that $(\boldsymbol{E'})$ is satisfied. $\square$

As an application of the lemmas we have just proved, we now give an example of an important class of weight functions for which $(\boldsymbol{E'})$ is satisfied for many kernels k.

Example 3.8. *Choose $\alpha \in (0,1)$, $a \ge 0$ and $p,q \in \mathbb{R}$ and define*

$$w(s) = \exp(as^{\alpha})(1+s)^p\big(\ln(e+s)\big)^q, \quad s \in \mathbb{R}_+.$$

Moreover, assume α, a, p, q are such that $w^{-1} \in L^1(\mathbb{R}_+)$ (i.e. $a > 0$ or $p > 1$ or $p = 1$ and $q > 1$) and (1.4) holds. Then

$$\ln w(s) = as^{\alpha} + p\ln(1+s) + q\ln\ln(e+s),$$

so that

$$\frac{w'(s)}{w(s)} = \frac{d}{ds}\ln w(s) = a\alpha s^{\alpha-1} + \frac{p}{1+s} + \frac{q}{(e+s)\ln(e+s)}$$

and

$$\frac{d}{ds}\frac{w'(s)}{w(s)} = -a\alpha(1-\alpha)s^{\alpha-2} - \frac{p}{(1+s)^2} - \frac{q}{(e+s)^2\ln(e+s)} - \frac{q}{(\ln(e+s))^2(e+s)^2} \le 0,$$

for all sufficiently large s. Thus, if

$$\int_s^{s+1}|\kappa(t)|\,dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } s\to\infty,$$

the assumptions of Proposition 3.7 (in case $a \ne 0$) and Proposition 3.2 (in case $a = 0$) are satisfied, so that $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ hold.

The following proposition can be seen as a generalization of the second case of Proposition 3.2.

Proposition 3.9. *Suppose that k satisfies $(\boldsymbol{A'})$ with $\kappa \in L^1(\mathbb{R})$. Assume further that $g \in C^1(0,\infty)$ satisfies*

$$g(s) > 0, \quad 0 < \frac{g'(s)}{g(s)} \leq \frac{1}{s}, \quad \text{for } s > 0, \tag{3.30}$$

and that

$$g(s)\frac{w'(s)}{w(s)} = O(1), \quad w(s)\int_{g(s)}^{\infty} |\kappa(t)|\, dt = O(1), \tag{3.31}$$

as $s \to \infty$. Then Assumptions $(\boldsymbol{E'})$ and $(\boldsymbol{F'})$ are satisfied.

Proof. Note that (3.30) implies that

$$1 \leq \frac{g(s)}{g(t)} \leq \frac{s}{t}, \quad 0 < t \leq s. \tag{3.32}$$

Note also that the second equation in (3.31) implies that $g(s) \to \infty$ as $s \to \infty$ and that the first of equations (3.31) implies that, for some $C > 0$ and all $s \geq t$ with t sufficiently large,

$$\frac{w(s)}{w(t)} = \exp\left(\int_t^s \frac{w'(u)}{w(u)}\, du\right) \leq \exp\left(\frac{C(s-t)}{g(t)}\right), \tag{3.33}$$

so that $(\boldsymbol{F'})$ holds.

Let us now first suppose that for some $\theta \in (0,1)$ the inequality $g(s) \leq \theta s$ is true for all sufficiently large s. It follows from (3.32) and the inequality (3.33) that, for all sufficiently large s,

$$\frac{w(s)}{w(s-g(s))} \leq \exp\left(\frac{Cg(s)}{g(s-g(s))}\right) \leq \exp\left(\frac{Cg(s)}{g((1-\theta)s)}\right) \leq \exp\left(\frac{C}{1-\theta}\right).$$

Thus, for sufficiently large A and all $s \geq 2A$,

$$\int_{\min\{s-g(s),s-A\}}^{s-A} \frac{w(s)}{w(t)}|\kappa(s-t)|\, dt \leq \exp\left(\frac{C}{1-\theta}\right)\int_A^{\infty} |\kappa(t)|\, dt,$$

while

$$\int_A^{s-g(s)} \frac{w(s)}{w(t)}|\kappa(s-t)|\, dt \leq \frac{w(s)}{w(A)}\int_{g(s)}^{\infty} |\kappa(t)|\, dt.$$

Combining the last two inequalities and noting (3.31) we see that $(\boldsymbol{E'})$ must be satisfied.

If it is not true that for some $\theta > 0$ the inequality $g(s) \leq \theta s$ holds for all sufficiently large s, then there exist sequences $\theta_n \to 1$ and $s_n \to \infty$ such that $g(s_n) \geq \theta_n s_n$. From (3.32) it follows that $g(t) \geq t g(s_n)/s_n \geq \theta_n t$, $0 < t \leq s_n$, and hence that $g(t) \geq t$, $t > 0$, so that (3.6) holds for some $\theta > 0$ and all sufficiently large s. But also, from (3.32), $g(s) \leq g(1)s$, $s \geq 1$. Thus $g(s/g(1)) \leq s$, for $s \geq g(1)$, and so, by (3.31),

$$w\Big(\frac{s}{g(1)}\Big)\int_s^{\infty} |\kappa(t)|\, dt \leq w\Big(\frac{s}{g(1)}\Big)\int_{g(s/g(1))}^{\infty} |\kappa(t)|\, dt = O(1), \quad \text{as } s \to \infty.$$

Further, from (3.33) and since $g(s/g(1)) \geq s/g(1)$, for $s > 0$, it holds that $w(s)/w(s/g(1)) \leq \exp(C(g(1)-1))$ for all sufficiently large s. Thus (3.8) holds. It follows from Proposition 3.2 that $\boldsymbol{(E')}$ is satisfied. $\square$

We now use this lemma to show that, for every kernel k satisfying $\boldsymbol{(A')}$ with $\kappa \in L^1(\mathbb{R})$, there exists a weight function w such that Assumption $\boldsymbol{(E')}$ holds. The construction is based on [4, p.58].

Suppose we are given a kernel k which satisfies $\boldsymbol{(A')}$ with $\kappa \in L^1(\mathbb{R})$. Then, provided $\mu(s) > 0$ for all $s \in \mathbb{R}_+$, a first guess at such a weight function might be $w(s) := \mu(0)/\mu(s)$, $s \in \mathbb{R}_+$, where

$$\mu(s) := \int_s^\infty |\kappa(t)|\,dt, \quad s \in \mathbb{R}_+. \tag{3.34}$$

Then, at least for almost all $s \in \mathbb{R}_+$ (or even for all $s \in \mathbb{R}_+$ if κ is continuous), the derivative $w'(s)$ exists and $w'(s) = |\kappa(s)|/\mu(s)^2$, so that Proposition 3.2 shows that $\boldsymbol{(E')}$ holds if

$$\frac{sw'(s)}{w(s)} = \frac{s|\kappa(s)|}{\mu(s)} = O(1), \quad s \to \infty. \tag{3.35}$$

Alternatively, if, for some $\alpha \in (0,1)$,

$$\frac{s^\alpha|\kappa(s)|}{\mu(s)} = O(1), \quad s \to \infty, \quad \text{and} \quad \frac{s|\kappa(s)|}{\mu(s)} \to \infty, \quad s \to \infty, \tag{3.36}$$

and $w'(s)/w(s)$ is monotonic increasing for all sufficiently large s, then Proposition 3.7 implies that $\boldsymbol{(E')}$ holds.

Conditions (3.35) and (3.36) contain rather strong pointwise estimates of κ. It therefore makes sense to introduce some averaging process in the definition of w. We also augment the definition of w to make the point that, given any $y \in X_0 := \{x \in X : x(s) \to 0 \text{ as } s \to \infty\}$, we can construct w such that $y \in X_w$. Let $y \in X_0 \setminus \{0\}$ and, for some $\beta \in (0,1)$,

$$v(s) := \min\left\{\frac{\mu(0)}{\mu(s^\beta)}, \frac{\|y\|}{\sup_{t\geq s}|y(t)|}, (1+s)^{1-\beta}\right\}, \quad s \in \mathbb{R}_+, \tag{3.37}$$

and define $w \in C(\mathbb{R}_+) \cap C^1(0,\infty)$ by

$$w(s) := \begin{cases} 1, & s = 0, \\ \frac{2}{s}\int_{s/2}^s v(t)\,dt, & s > 0. \end{cases} \tag{3.38}$$

Note that

$$(1+s)^{1-\beta} \geq v(s) \geq w(s) \geq v(s/2) \geq 1, \quad s \geq 0.$$

We also have that

$$w'(s) = \frac{2v(s) - v(s/2) - w(s)}{s} \geq 0, \quad s > 0.$$

Thus (1.4) holds and

$$\frac{w'(s)}{w(s)} = \frac{2v(s) - v(s/2) - w(s)}{sw(s)} \leq \frac{2v(s)}{s} \leq 2\frac{(1+s)^{1-\beta}}{s}, \quad s > 0.$$

Thus, setting $g(s) := s^\beta$, $g(s)w'(s)/w(s) = O(1)$ as $s \to \infty$ and

$$w(s)\mu(g(s)) \leq v(s)\mu(s^\beta) \leq \mu(0),$$

so that our last proposition applies. Further, for $s \in \mathbb{R}_+$, $|y(s)|w(s) \leq |y(s)|v(s) \leq \|y\|$, so that $y \in X_w$. We have thus obtained the following theorem.

Theorem 3.10. *Suppose the kernel k satisfies **(A′)** with $\kappa \in L^1(\mathbb{R})$ and $y \in X_0$. Then there exists a weight function w, defined by equations* (3.34), (3.37) *and* (3.38), *satisfying the conditions* (1.4) *and Assumption **(F′)** and such that Assumption **(E′)** holds and $y \in X_w$.*

As an interesting consequence of this result we relate the solvability of (1.1) in X_0, with the same norm $\|\cdot\|$ a closed subspace of X and so a Banach space in its own right, to its solvability in X in the following theorem. With the additional assumption that $\kappa(s) = O(s^{-q})$ as $s \to \infty$ for some $q > 1$ this result has been shown previously in [4].

Theorem 3.11. *Suppose that k satisfies **(A′)**, with $\kappa \in L^1(\mathbb{R})$, and **(B)**. Then $K \in B(X) \cap B(X_0)$ and*

$$\Sigma_{X_0}(K) \subset \Sigma_X(K).$$

Proof. By Theorem 3.10, given any $y \in X_0$ there exists $w = w(y)$ such that (1.4), **(E′)** and **(F′)** hold and $y \in X_w$. From Lemma 2.3 it follows that $Ky \in X_w \subset X_0$. Thus, and since we have $\|Kx\| \leq \|\kappa\|_1\|x\|$ for all $x \in X$, it holds that $K \in B(X) \cap B(X_0)$.

Now, suppose that $\lambda \notin \Sigma_X(K)$. Then, by Theorem 2.7, for every $y \in X_0$, $\lambda \notin \Sigma_{X_{w(y)}}(K)$. In particular, for every $y \in X_0$, it follows that there exists $x \in X_{w(y)} \subset X_0$ such that $(\lambda - K)x = y$, so that $\lambda - K : X_0 \to X_0$ is surjective. Moreover, $\lambda - K$ is injective on $X_0 \subset X$ since it is injective on X. Thus $\lambda \notin \Sigma_X(K)$ implies that $(\lambda - K) : X_0 \to X_0$ is bijective and hence, since X_0 is a Banach space, that $(\lambda - K)^{-1} \in B(X_0)$, i.e. that $\lambda \notin \Sigma_{X_0}(K)$. □

4. The real line case

All our results in Sections 2 and 3 are concerned with integral operators defined on the half line $\mathbb{R}_+$. However, many practical applications lead to integral equations on the real line. In particular we mention recent work on boundary integral equations for scattering by infinite rough surfaces, see [7, 1] and the references therein. To indicate how our results carry over to the real line case, let us now state modified versions of the main results for the real line integral equation

$$\lambda x(s) - \int_{-\infty}^{\infty} k(s,t)x(t)\,dt = y(s), \quad s \in \mathbb{R},$$

where $k : \mathbb{R}^2 \to \mathbb{C}$ and $x, y \in X := BC(\mathbb{R})$, the space of bounded and continuous functions on $\mathbb{R}$. We note that, as discussed in the introduction, the results stated below, in particular Theorem 4.2 and Propositions 4.3-4.5, are applied in [6, 17] to obtain error estimates for the finite section method when the kernel takes the form $k(s,t) = \kappa(s-t)z(t)$, with $\kappa \in L^1(\mathbb{R})$ and $z \in L^\infty(\mathbb{R})$.

We define the integral operator K by

$$Kx(s) := \int_{-\infty}^{\infty} k(s,t)x(t)\,dt, \quad s \in \mathbb{R},$$

in analogy to the definition (1.2). The replacements for $\boldsymbol{(A)}$, $\boldsymbol{(A')}$ and $\boldsymbol{(B)}$ we need to impose on K are

$$(\boldsymbol{A}_{\mathbb{R}}) \qquad \sup_{s\in\mathbb{R}} \int_{-\infty}^{\infty} |k(s,t)|\,dt < \infty,$$

$$(\boldsymbol{B}_{\mathbb{R}}) \qquad \forall s \in \mathbb{R} \int_{-\infty}^{\infty} |k(s,t) - k(s',t)|\,dt \to 0 \quad \text{as } s' \to s,$$

$$(\boldsymbol{A}'_{\mathbb{R}}) \qquad |k(s,t)| \le |\kappa(s-t)|, \quad s,t \in \mathbb{R}.$$

Throughout this section let $w \in C(\mathbb{R})$ be an even function such that the restriction of w to $\mathbb{R}_+$ satisfies (1.4). We consider real-line variants of the spaces X and X_w, namely $X := \{x \in C(\mathbb{R}) : \|x\| < \infty\}$ and $X_w := \{x \in C(\mathbb{R}) : \|x\|_w < \infty\}$, where

$$||x|| := \sup_{s\in\mathbb{R}} |x(s)|, \quad ||x||_w := \sup_{s\in\mathbb{R}} |w(s)x(s)|.$$

Similarly to the half-line case, $(\boldsymbol{A}_{\mathbb{R}})$ and $(\boldsymbol{B}_{\mathbb{R}})$ ensure that K is a bounded operator on X. Towards boundedness in X_w we can use the symmetry of w to obtain the following variant of Proposition 2.1 and Corollary 2.2.

Proposition 4.1. *Suppose that the kernel k satisfies Assumptions $(\boldsymbol{A}'_{\mathbb{R}})$ and $(\boldsymbol{B}_{\mathbb{R}})$, with $\kappa \in L^1(\mathbb{R})$ in $(\boldsymbol{A}'_{\mathbb{R}})$. Then $K \in B(X_w)$ if*

$$\int_{-|s|}^{|s|} \frac{|\kappa(s-t)|}{w(t)}\,dt = \int_{s-|s|}^{s+|s|} \frac{|\kappa(t)|}{w(s-t)}\,dt = O\Big(\frac{1}{w(s)}\Big), \quad \text{as } |s| \to \infty, \tag{4.1}$$

in which case $k_w(s,t) := (w(s)/w(t))k(s,t)$ satisfies $(\boldsymbol{A}_{\mathbb{R}})$ and $(\boldsymbol{B}_{\mathbb{R}})$. If $k(s,t) = \kappa(s-t)$ for some $\kappa \in L^1(\mathbb{R})$, then $K \in B(X_w)$ if and only if (4.1) holds.

In the remainder of this section let, for $s \ge 0$,

$$\lambda(s) := \int_s^{s+1} |\kappa(t)|\,dt + \int_{-s-1}^{-s} |\kappa(t)|\,dt,$$

and

$$\mu(s) := \int_{\mathbb{R}\setminus[-s,s]} |\kappa(t)|\,dt = \int_s^{\infty} |\kappa(t)|\,dt + \int_{-\infty}^{-s} |\kappa(t)|\,dt.$$

Arguing as in Section 2, if $\kappa \neq 0$ then it follows from (4.1) that $\boldsymbol{(F)}$ holds. And if $\boldsymbol{(F)}$ holds we note that (4.1) implies that

$$w(s)\lambda(s) = O(1) \quad \text{as } s \to \infty.$$

The key theorem 2.4 remains valid in the real line case if we replace $\boldsymbol{(A)}$, $\boldsymbol{(B)}$ and $\boldsymbol{(D)}$ by their real line versions, the real line version of $\boldsymbol{(D)}$ being

$$(\boldsymbol{D}_{\mathbb{R}}) \qquad \sup_{s\in\mathbb{R}} \int_{\mathbb{R}\setminus[-A,A]} |k(s,t)|\,dt \to 0 \quad \text{as } A \to \infty.$$

Introducing the condition

$$(\boldsymbol{E}'_{\mathbb{R}}) \qquad \sup_{|s|\geq 2A} \left(\int_{-|s|+A}^{-A} + \int_{A}^{|s|-A} \right) \frac{w(s)}{w(t)} |\kappa(s-t)|\,dt \to 0, \qquad \text{as } A \to \infty,$$

Theorem 2.5 and Corollary 2.6 remain valid in the real line case, with Assumptions $\boldsymbol{(A)}$, $\boldsymbol{(B)}$, $\boldsymbol{(D)}$, and $\boldsymbol{(E')}$ replaced by $(\boldsymbol{A}_{\mathbb{R}})$, $(\boldsymbol{B}_{\mathbb{R}})$, $(\boldsymbol{D}_{\mathbb{R}})$, and $(\boldsymbol{E}'_{\mathbb{R}})$, respectively. (Since w is assumed even, we do not need to modify $\boldsymbol{(F')}$.) Thus we can prove the following variant of the main result of Section 2, Theorem 2.7, by using the same argument as in the proof of that theorem.

Theorem 4.2. *Suppose that k and w satisfy $(\boldsymbol{A}'_{\mathbb{R}})$, $(\boldsymbol{B}_{\mathbb{R}})$, $(\boldsymbol{E}'_{\mathbb{R}})$ and $\boldsymbol{(F')}$, with $\kappa \in L^1(\mathbb{R})$ in $(\boldsymbol{A}'_{\mathbb{R}})$. Then, for any $\lambda \in \mathbb{C}$,*

$$(\lambda - K) \in \Phi(X) \Leftrightarrow (\lambda - K_w) \in \Phi(X) \Leftrightarrow (\lambda - K) \in \Phi(X_w),$$

and if these operators are Fredholm, their indices coincide. Thus

$$\Sigma^e_X(K) = \Sigma^e_X(K_w) = \Sigma^e_{X_w}(K)$$

and it holds, moreover, that

$$\Sigma_X(K) = \Sigma_X(K_w) = \Sigma_{X_w}(K).$$

We finish the paper by stating results which specify simpler conditions on w and k that ensure that the conditions of Theorem 4.2 are satisfied. We start with a variant of Propositions 3.2 and 3.3.

Proposition 4.3. *Suppose that k satisfies $(\boldsymbol{A}'_{\mathbb{R}})$, with $\kappa \in L^1(\mathbb{R})$, and that there exists $\theta > 0$ such that for all sufficiently large $s > 0$ the inequality*

$$\frac{w'(s)}{w(s)} \leq \frac{\theta}{s} \tag{4.2}$$

holds. Further, suppose that either

$$w^{-1} \in L^1(\mathbb{R}) \quad \text{and} \quad \lambda(s) = O\Big(\frac{1}{w(s)}\Big), \quad s \to \infty,$$

or, alternatively,

$$w(s)\mu(s) = O(1), \quad s \to \infty,$$

holds. Then Assumptions $(\boldsymbol{E}'_{\mathbb{R}})$ and $\boldsymbol{(F')}$ are satisfied.

If w satisfies (4.2), for some $\theta \leq 1$ and all sufficiently large $s > 0$, and

$$\lambda(s) = \begin{cases} O(s^{-1}) & , \text{ if } \theta < 1, \\ o((s \ln s)^{-1}) & , \text{ if } \theta = 1, \end{cases} \quad \text{as } s \to \infty,$$

then k satisfies $(\boldsymbol{E}'_{\mathbb{R}})$.

The proof of this proposition can, with slight modifications, be read off the corresponding proofs in Section 3. The same is true for the final two propositions, which are the real-line variants of Propositions 3.7 and 3.9.

Proposition 4.4. *Suppose w fulfils the conditions of Proposition 3.7 and that k satisfies $(\boldsymbol{A'_{\mathbb{R}}})$, with $\kappa \in L^1(\mathbb{R})$, and also*

$$\lambda(s) = O\Big(\frac{1}{w(s)}\Big), \quad s \to \infty.$$

Then Assumptions $(\boldsymbol{E'_{\mathbb{R}}})$ and $(\boldsymbol{F'})$ are satisfied.

Proposition 4.5. *Suppose that k satisfies $(\boldsymbol{A'_{\mathbb{R}}})$ with $\kappa \in L^1(\mathbb{R})$. Assume further that $g \in C^1(0,\infty)$ is a positive function which satisfies condition* (3.30) *of Proposition 3.9. Moreover, assume that*

$$g(s)\frac{w'(s)}{w(s)} = O(1), \quad w(s)\mu(g(s)) = O(1),$$

as $s \to \infty$. Then Assumptions $(\boldsymbol{E'_{\mathbb{R}}})$ and $(\boldsymbol{F'})$ are satisfied.

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