The mathematics of scattering by unbounded, rough, inhomogeneous layers ⋆

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Abstract

In this paper we study, via variational methods, a boundary value problem for the Helmholtz equation modelling scattering of time harmonic waves by a layer of spatially-varying refractive index above an unbounded rough surface on which the field vanishes. In particular, in the 2D case with TE polarization, the boundary value problem models the scattering of time harmonic electromagnetic waves by an inhomogeneous conducting or dielectric layer above a perfectly conducting unbounded rough surface, with the magnetic permeability a fixed positive constant in the medium. Via analysis of an equivalent variational formulation, we show that this problem is well-posed in two important cases: when the frequency is small enough; and when the medium in the layer has some energy absorption. In this latter case we also establish exponential decay of the solution with depth in the layer. An attractive feature is that all constants in our estimates are bounded by explicit functions of the index of refraction and the geometry of the scatterer.

Key words: Non-smooth boundary, radiation condition, a priori estimate
PACS: 35J05, 35J20, 35J25, 42B10, 78A45

1 Introduction

This paper is concerned with the rigorous study of a class of *rough surface scattering problems*. We use the phrase *rough surface* to denote a surface which

⋆ This work was supported by the UK Engineering and Physical Sciences Research Council under Grant GR/S67401. The research of Peter Monk is also supported by AFOSR grant F49620-02-1-0071.

is a perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane.

Rough surface scattering problems arise frequently in applications, for example in modelling acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces, and are widely studied in the engineering literature, with a view to developing both rigorous methods of computation and approximate, asymptotic, or statistical methods (see Voronovich [1], Saillard & Sentenac [2], Warnick & Chew [3], DeSanto [4], Reitich and Turc [5], and the references therein).

This paper is concerned with the rigorous derivation of variational formulations for problems of this type, and with establishing the well-posedness of these variational formulations, under appropriate constraints on the geometry and the medium of propagation. We consider a specific two- or three-dimensional rough surface scattering problem which models time harmonic acoustic scattering ($\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence) by a layer of inhomogeneous wave speed above a sound soft rough surface. The same mathematics, in the 2D case, models time harmonic electromagnetic scattering by an inhomogeneous conducting or dielectric layer above a perfectly conducting surface in the case of TE (transverse electric) polarization. Thus, we will seek to solve the inhomogeneous Helmholtz equation with space-dependent wave number k, i.e.

$$\Delta u + k^2 u = g,$$

in the perturbed half-plane or half-space $D \subset \mathbb{R}^n$, $n = 2, 3$. In the acoustics case, $k = \omega/c$ with c the spatially varying wave speed. In the electromagnetic case of TE polarization u denotes the component of the electric field that is perpendicular to the plane and

$$k^2 = \omega^2 \mu \epsilon [1 + \mathrm{i}\sigma/(\omega\epsilon)], \tag{1}$$

where $\epsilon > 0$ is the electric permittivity, $\sigma \geq 0$ is the conductivity, both of which we suppose may be spatially varying, and $\mu > 0$ is the magnetic permeability, which we suppose to be constant. We assume, moreover, that the variation in k is confined to a neighbourhood of ∂D. Choosing a coordinate system so that $\partial D \subset \{x = (x_1, \ldots, x_n) : f_- < x_n < f_+\}$, for some $f_- < f_+$, with the upper half-plane or half-space $\{x : x_n > f_+\}$ a part of D, we assume that $k(x) = k_+$ whenever $x_n > H$, for some constants $k_+ > 0$ and $H > f_+$. We suppose that the homogeneous Dirichlet boundary condition $u = 0$ holds on ∂D, which corresponds in the electromagnetic case to ∂D being perfectly conducting. We will impose a suitable radiation condition to ensure that the field u is outgoing in an appropriate sense. We give in the next section complete details about our assumptions on D and k and about the radiation condition we impose.

The main results of the paper are the following. In the next section we formulate the boundary value problem precisely, in the case when $g \in L^2(D)$ with support lying within a finite distance of ∂D. We also establish the equivalent variational formulation that we use and study in this paper. As part of the boundary value problem formulation we require the radiation condition often used in a formal manner in the rough surface scattering literature (e.g. [4]), that, above the rough surface and the support of g, the solution can be represented in integral form as a superposition of upward traveling and evanescent plane waves. This radiation condition is equivalent to the upward propagating radiation condition proposed for two-dimensional rough surface scattering problems in [6], and has recently been analyzed carefully in the 2D case by Arens and Hohage [7].

In Section 3 we analyze the variational formulation in two cases in which it emerges that the sesquilinear form is elliptic, so that unique existence of solution and explicit bounds on the solution in terms of the data g follow from the Lax-Milgram lemma. These cases are: (i) the case of low frequency (ω small); (ii) the case when the medium is energy-absorbing in $x_n < H$, satisfying that the argument of $k^2(x)$ is bounded away from zero.

In the final section we show that the sesquilinear form remains elliptic when considered as a sesquilinear form on certain weighted spaces. This observation leads to explicit bounds on the exponential decay of the solution as x_n decreases in the case that the imaginary part of k is bounded away from zero.

This paper is closest to a recent study by Chandler-Wilde and Monk [8] who consider the same mathematical and physical problem, but restricted to the case in which $k(x) \equiv k_+$ is a positive constant in the whole domain D. In this current paper we make a partial extension, as outlined above, to the cases when k is variable and/or complex-valued in D. Both of these extensions are of significant practical importance (for example, the 2D electromagnetic situation modelled, of diffraction by a dielectric/conducting layer, is extensively studied in diffractive optics, e.g. [9]).

2 The boundary value problem and variational formulation

In this section we introduce the boundary value problem and its equivalent variational formulation that will be analyzed in later sections. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ $(n = 2, 3)$ let $\tilde{x} = (x_1, \ldots, x_{n-1})$ so that $x = (\tilde{x}, x_n)$. For $H \in \mathbb{R}$ let $U_H = \{x : x_n > H\}$ and $\Gamma_H := \{x : x_n = H\}$. Let $D \subset \mathbb{R}^n$ be a connected open set such that for some constants $f_- < f_+$ it holds that

$$U_{f_+} \subset D \subset U_{f_-}, \tag{2}$$

and let $\Gamma = \partial D$ denote the boundary of ∂D (the rough surface). The variational problem will be posed on the open set $S_H := D \setminus \overline{U}_H$, for some $H \geq f_+$.

Given a source $g \in L^2(D)$ whose support lies within a finite distance of the boundary, and given $k \in L^\infty(D)$, such that $k(x) = k_+$, $x \in U_H$, for some $k_+ > 0$ and $H \geq f_+$, the problem we wish to analyze is to find the field u such that

$$\Delta u + k^2 u = g \text{ in } D, \tag{3}$$

$$u = 0 \text{ on } \Gamma, \tag{4}$$

and such that u satisfies an appropriate radiation condition.

This problem has recently been studied in a rigorous manner using variational methods by Chandler-Wilde & Monk [8] in the special case in which k is a positive constant in the whole domain D; indeed this paper will be a main starting point for the methods and arguments we make, the contribution of the present paper being to extend the results of [8] to the case where k varies as a function of position. This extension is non-trivial. Indeed, the case when k is variable has been extensively studied, but to date only for the 2D case and, for the most part, for the simpler case of a diffraction grating, where the problem is to compute the scattered field when a plane wave is incident and the geometry is periodic, so that $k(x + Le_1) = k(x)$, $x \in D$, for some constant $L > 0$, where e_1 is the unit vector in the x_1-direction. This is the subject, in particular, of the mathematical studies of Bonnet-Bendhia & Starling [10], Strycharz-Szemberg [11], and Elschner & Schmidt [12], who consider this problem as a model of electromagnetic scattering in TE polarization (the electric field perpendicular to the 2D plane) when the diffraction grating is penetrable, with variable permittivity and conductivity.

The diffraction grating problem is simpler since the variational formulation is on a single periodic cell, a compact set, as a consequence of which the sesquilinear form satisfies a Gårding inequality, so that the associated linear operator is Fredholm of index zero and well-posedness follows from uniqueness.

The 2D version of our problem, without an assumption of periodicity of k, has been considered by integral equation methods, but only in two cases. The first is the case that k is constant and Γ is the graph of a sufficiently smooth bounded function f, when boundary integral equation methods are applicable [13,14]. The second case studied is that in which $\Gamma = \partial D$ is a straight line, so that D is a half-plane; this case is reduced to a Lippmann-Schwinger-type integral equation in [15]. These papers establish existence of solution is certain cases by a partial generalization of the Riesz-Fredholm theory of compact operators to the case where the operator is only locally compact [16], so that existence can be deduced from uniqueness of solution. In this paper

we will establish existence of solution in certain cases via application of the Lax Milgram lemma to a carefully chosen sesquilinear form.

To complete the formulation of our boundary value problem we need a radiation condition, and we make use of the radiation condition employed in [8]. For $\phi \in L^2(\Gamma_H)$, which we identify with $L^2(\mathbb{R}^{n-1})$, we denote by $\hat{\phi} = \mathcal{F}\phi$ the Fourier transform of ϕ which we define by

$$\mathcal{F}\phi(\xi) = (2\pi)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} \exp(-\mathrm{i}\tilde{x}\cdot\xi)\phi(\tilde{x})\, d\tilde{x}, \quad \xi \in \mathbb{R}^{n-1}. \tag{5}$$

Our *radiation condition* is to require that

$$u(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \exp(\mathrm{i}[(x_n - H)\sqrt{k_+^2 - \xi^2} + \tilde{x}\cdot\xi])\hat{F}_H(\xi)\, d\xi,\ x \in U_H, \tag{6}$$

where $F_H := u|_{\Gamma_H} \in L^2(\Gamma_H)$. In this equation $\sqrt{k_+^2 - \xi^2} = \mathrm{i}\sqrt{\xi^2 - k_+^2}$, when $|\xi| > k_+$.

Equation (6) is a representation for u, in the upper half-plane U_H, as a superposition of upward propagating homogeneous and inhomogeneous plane waves. A requirement that (6) holds is commonly used (e.g. [4]) as a formal radiation condition in the physics and engineering literature on rough surface scattering. The meaning of (6) is clear when $F_H \in L^2(\mathbb{R}^{n-1})$ so that $\hat{F}_H \in L^2(\mathbb{R}^{n-1})$; indeed the integral (6) exists in the Lebesgue sense for all $x \in U_H$. Recently Arens and Hohage [7] have explained, in the case $n = 2$, in what precise sense (6) can be understood when $F_H \in BC(\Gamma_H)$, the space of bounded continuous functions on Γ_H, so that $\hat{F}_H$ must be interpreted as a tempered distribution.

We now precisely state our boundary value problem. Let $H_0^1(D)$ denote the standard Sobolev space, the completion of $C_0^\infty(D)$ in the norm $\|\cdot\|_{H^1(D)}$ defined by $\|u\|_{H^1(D)} = \{\int_D (|\nabla u|^2 + |u|^2)dx\}^{1/2}$. The main function space in which we set our problem will be the Hilbert space V_H, defined, for $H \geq f_+$, by $V_H := \{\phi|_{S_H} \ : \ \phi \in H_0^1(D)\}$, on which we will impose a wave number dependent scalar product $(u, v)_{V_H} := \int_{S_H} (\nabla u \cdot \overline{\nabla v} + k_+^2 u\bar{v})\, dx$ and norm, $\|u\|_{V_H} = \{\int_{S_H} (|\nabla u|^2 + k_+^2|u|^2)dx\}^{1/2}$.

The boundary value problem. *Given $g \in L^2(D)$, and $k \in L^\infty(D)$ such that for some $H \geq f_+$ it holds that the support of g lies in $\overline{S_H}$ and that $k(x) = k_+$, $x \in \overline{U_H}$, for some $k_+ > 0$, find $u : D \to \mathbb{C}$ such that $u|_{S_a} \in V_a$ for every $a > f_+$, $\Delta u + k^2 u = g$ in D in a distributional sense, and the radiation condition (6) holds, with $F_H = u|_{\Gamma_H}$.*

Remark 1 *We note that, as one would hope (see [8, Remark 2.1]), the solu-*

tions of the above problem do not depend on the choice of H. Precisely, if u is a solution to the above problem for one value of $H \geq f_+$ for which $\operatorname{supp} g \subset \overline{S_H}$ *and $k = k_+$ in $\overline{U_H}$, then u is a solution for all $H \geq f_+$ with this property.*

We now derive a variational formulation of the boundary value problem above, in which trace operators and a Dirichlet-to-Neumann operator play a role. To describe the mapping properties of these operators we will use standard fractional Sobolev space notation, except that we adopt a wave number dependent norm, equivalent to the usual norm, and reducing to the usual norm if the unit of length measurement is chosen so that $k_+ = 1$. Thus, identifying Γ_H with $\mathbb{R}^{n-1}$, $H^s(\Gamma_H)$, for $s \in \mathbb{R}$, denotes the completion of $C_0^\infty(\Gamma_H)$ in the norm $\|\cdot\|_{H^s(\Gamma_H)}$ defined by

$$\|\phi\|_{H^s(\Gamma_H)} = \left(\int_{\mathbb{R}^{n-1}} (k_+^2 + \xi^2)^s |\mathcal{F}\phi(\xi)|^2 \, d\xi \right)^{1/2}.$$

We recall [17] that, for all $a > H \geq f_+$, there exist continuous embeddings $\gamma_+ : H^1(U_H \setminus U_a) \to H^{1/2}(\Gamma_H)$ and $\gamma_- : V_H \to H^{1/2}(\Gamma_H)$ (the trace operators) such that $\gamma_\pm \phi$ coincides with the restriction of ϕ to Γ_H when ϕ is C^∞. In the case when $H = f_+$, when Γ_H may not be a subset of the boundary of S_H (if part of ∂D coincides with Γ_H) we understand this trace by first extending $\phi \in V_H$ by zero to $U_{f_-} \setminus \overline{U_{f_+}}$. We recall also that, if $u_+ \in H^1(U_H \setminus U_a)$, $u_- \in V_H$, and $\gamma_+ u_+ = \gamma_- u_-$, then $v \in V_a$, where $v(x) := u_+(x)$, $x \in U_H \setminus U_a$, $:= u_-(x)$, $x \in S_H$. Conversely, if $v \in V_a$ and $u_+ := v|_{U_H \setminus U_a}$, $u_- := v|_{S_H}$, then $\gamma_+ v_+ = \gamma_- v_-$.

It is easy to see that, if $F_H \in C_0^\infty(\Gamma_H)$ and u is given by (6), then

$$T\gamma_+ u = -\left. \frac{\partial u}{\partial x_n} \right|_{\Gamma_H}, \tag{7}$$

where the Dirichlet to Neumann map T is defined by

$$T := \mathcal{F}^{-1} M_z \mathcal{F}, \tag{8}$$

with M_z the operation of multiplying by $z(\xi) := \sqrt{\xi^2 - k_+^2}$, $\xi \in \mathbb{R}$, where we take the square root with negative imaginary part, $z(\xi) = -\mathrm{i}\sqrt{k_+^2 - \xi^2}$, for $|\xi| \leq k_+$. It is shown in [8, Lemma 2.5] that $T : H^{1/2}(\Gamma_H) \to H^{-1/2}(\Gamma_H)$ and is bounded, with norm

$$\|T\| = 1. \tag{9}$$

We recall the following lemma from [8], which describes properties of u, defined by (6).

Lemma 2 *If (6) holds, with $F_H \in H^{1/2}(\Gamma_H)$, then $u \in H^1(U_H \setminus U_a) \cap C^2(U_H)$, for every $a > H$,*

$$\Delta u + k_+^2 u = 0 \text{ in } U_H,$$

$\gamma_+ u = F_H$, *and*

$$\int_{\Gamma_H} \bar{v} T \gamma_+ u \, ds + k_+^2 \int_{U_H} u \bar{v} \, dx - \int_{U_H} \nabla u \cdot \nabla \bar{v} \, dx = 0, \quad v \in C_0^\infty(D). \tag{10}$$

Now suppose that u satisfies the boundary value problem. Then $u|_{S_a} \in V_a$ for every $a > f_+$ and, by definition, since $\Delta u + k^2 u = g$ in a distributional sense,

$$\int_D [g\bar{v} + \nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}] dx = 0, \quad v \in C_0^\infty(D). \tag{11}$$

Applying Lemma 2, and defining $w := u|_{S_H}$, it follows that

$$\int_{S_H} [g\bar{v} + \nabla w \cdot \nabla \bar{v} - k^2 w \bar{v}] \, dx + \int_{\Gamma_H} \bar{v} T \gamma_- w \, ds = 0, \quad v \in C_0^\infty(D).$$

From the denseness of $\{\phi|_{S_H} : \phi \in C_0^\infty(D)\}$ in V_H and the continuity of γ_-, it follows that this equation holds for all $v \in V_H$.

Let $\|\cdot\|_2$ and $(\cdot,\cdot)$ denote the norm and scalar product on $L^2(S_H)$, so that $\|v\|_2 = \sqrt{\int_{S_H} |v|^2 \, dx}$ and $(u,v) = \int_{S_H} u\overline{v} \, dx$, and define the sesquilinear form $b : V_H \times V_H \to \mathbb{C}$ by

$$b(u,v) = (\nabla u, \nabla v) - (k^2 u, v) + \int_{\Gamma_H} \gamma_- \bar{v} T \gamma_- u \, ds. \tag{12}$$

Then we have shown that if u satisfies the boundary value problem then $w := u|_{S_H}$ is a solution of the following variational problem: find $u \in V_H$ such that

$$b(u,v) = -(g,v), \quad v \in V_H. \tag{13}$$

Conversely, suppose that w is a solution to the variational problem and define $u(x)$ to be $w(x)$ in S_H and to be the right hand side of (6), with $F_H := \gamma_- w$, in U_H. Then, by Lemma 2, $u \in H^1(U_H \setminus U_a)$ for every $a > H$, with $\gamma_+ u = F_H = \gamma_- w$. Thus $u|_{S_a} \in V_a$, $a \geq f_+$. Further, from (10) and (13) it follows

that (11) holds, so that $\Delta u + k^2 u = g$ in D in a distributional sense. Thus u satisfies the boundary value problem.

We have thus proved the following theorem.

Theorem 3 *If u is a solution of the boundary value problem then $u|_{S_H}$ satisfies the variational problem. Conversely, if u satisfies the variational problem, $F_H := \gamma_- u$, and the definition of u is extended to D by setting $u(x)$ equal to the right hand side of (6), for $x \in U_H$, then the extended function satisfies the boundary value problem, with g extended by zero from S_H to D and k extended from S_H to D by taking the value k_+ in $\overline{U_H}$.*

3 V_H-Ellipticity of the sesquilinear form

In this section we shall investigate under what conditions b is V_H-elliptic. From the point of view of numerical solution by e.g. finite element methods, the ellipticity we establish is of course highly desirable, guaranteeing, by Céa's lemma, unique existence and stability of the numerical solution method.

Let V_H^* denote the dual space of V_H, i.e. the space of continuous anti-linear functionals on V_H. Then our analysis will also apply to the following slightly more general problem: given $\mathcal{G} \in V_H^*$ find $u \in V_H$ such that

$$b(u, v) = \mathcal{G}(v), \quad v \in V_H. \tag{14}$$

It will be assumed in the remainder of the paper that $k \in L^\infty(D)$ satisfies that $\Re(k^2) \geq 0, \Im(k^2) \geq 0$, which is certainly the case in the electromagnetic case where k^2 is given by (1). Then there exist constants $k_\infty \geq k_- \geq 0$ and $\theta \in [0, \pi/2]$ such that

$$k_- \leq |k(x)| \leq k_\infty, \quad \arg(k^2(x)) \geq \theta, \tag{15}$$

for almost all $x \in S_H$. It is convenient to introduce the dimensionless parameters

$$\kappa_\infty := k_\infty(H - f_-), \quad \kappa_- := k_-(H - f_-), \text{ and } \kappa_+ := k_+(H - f_-).$$

We shall prove the following theorem.

Theorem 4 *Suppose that either $\kappa_\infty < \sqrt{2}$ or $\theta > 0$. Then, for some constant $\alpha > 0$,*

$$|b(u, u)| \geq \alpha \|u\|_{V_H}^2, \quad u \in V_H,$$

so that the variational problem (14) is uniquely solvable. Moreover, the solution satisfies the estimate

$$\|u\|_{V_H} \leq C\|\mathcal{G}\|_{V_H^*} \tag{16}$$

where $C := \alpha^{-1}$ *satisfies*

$$C \leq \frac{2+\kappa_+^2}{2-\kappa_\infty^2}$$

if $\kappa_\infty < \sqrt{2}$*, and satisfies*

$$C \leq \operatorname{cosec}\theta \left(1 + \frac{\kappa_+^2}{\max(2,\kappa_-^2)}\right)$$

if $\theta > 0$*. In particular, the scattering problem (13) is uniquely solvable and the solution satisfies the bound*

$$k_+\|u\|_{V_H} \leq \max\left(1, \frac{\kappa_+}{\sqrt{2}}\right) C\|g\|_2. \tag{17}$$

Let us recall from [8] some results needed to prove Theorem 4.

Lemma 5 *For all* $u \in V_H$*,*

$$\|\gamma_- u\|_{H^{1/2}(\Gamma_H)} \leq \|u\|_{V_H} \text{ and } \|u\|_2 \leq \frac{H-f_-}{\sqrt{2}} \left\| \frac{\partial u}{\partial x_n} \right\|_2.$$

Lemma 6 *For all* $\phi, \psi \in H^{1/2}(\Gamma_H)$*,*

$$\int_{\Gamma_H} \phi T\psi ds = \int_{\Gamma_H} \psi T\phi ds.$$

For all $\phi \in H^{1/2}(\Gamma_H)$*,*

$$\Re \int_{\Gamma_H} \bar{\phi}\, T\phi\, ds \geq 0 \text{ and } \Im \int_{\Gamma_H} \bar{\phi}\, T\phi\, ds \leq 0.$$

The above lemmas imply that $b(\cdot,\cdot)$ is bounded, giving an explicit value for the bound, and that $b(\cdot,\cdot)$ has the following important symmetry property.

Corollary 7 *For all* $u, v \in V_H$*,* $b(v,u) = b(\bar{u},\bar{v})$*.*

Lemma 8 *For all* $u, v \in V_H$*,*

$$|b(u,v)| \leq \left[\frac{k_\infty^2}{k_+^2} + 1\right] \|u\|_{V_H}\|v\|_{V_H}$$

so that the sesquilinear form $b(.,.)$ is bounded.

Proof. From the definition of the sesquilinear form $b(.,.)$ and the Cauchy-Schwarz inequality we have

$$|b(u,v)| \leq \|\nabla u\|_2 \|\nabla v\|_2 + \frac{k_\infty^2 k_+^2}{k_+^2} \|u\|_2 \|v\|_2 + \|\gamma_- u\|_{H^{1/2}(\Gamma_H)} \|T\| \, \|\gamma_- v\|_{H^{1/2}(\Gamma_H)}.$$

Applying the Cauchy-Schwarz inequality, (9), and Lemma 5 we obtain the desired estimate. ∎

Our last lemma of this section shows that the sesquilinear form $b(.,.)$ is V_H-elliptic provided that κ_∞ is not too large or $\arg(k^2)$ is strictly positive.

Lemma 9 *i) For all $u \in V_H$,*

$$|b(u,u)| \geq \frac{2 - \kappa_\infty^2}{2 + \kappa_+^2} \|u\|_{V_H}^2.$$

ii) For all $u \in V_H$,

$$|b(u,u)| \geq \frac{\sin\theta}{1 + \kappa_+^2 / \max(2, \kappa_-^2)} \|u\|_{V_H}^2.$$

Proof. i) By Lemma 6, $\Re\, b(u,u) \geq \|u\|_{V_H}^2 - k_+^2 \|u\|_2^2 - k_\infty^2 \|u\|_2^2$. The result follows from Lemma 5 which implies that $\|u\|_{V_H}^2 \geq k_+^2 (2/\kappa_+^2 + 1) \|u\|_2^2$.

ii) Choose $\alpha \geq 0$ and define $\beta \in (0, \theta]$ by

$$\tan\beta = \frac{\sin\theta}{\alpha + \cos\theta},$$

so that $\alpha \sin\beta = \sin(\theta - \beta)$ and

$$\sin\beta = \frac{\sin\theta}{\sqrt{\alpha^2 + 2\alpha\cos\theta + 1}} \geq \frac{\sin\theta}{1+\alpha}.$$

Then, by Lemma 6, and since $\pi/2 - \beta \in [0, \pi/2]$,

$$\Re \left(\mathrm{e}^{i(\pi/2 - \beta)} \int_{\Gamma_H} \gamma_- \bar{u} T \gamma_- u \, ds \right) \geq 0.$$

Hence

$$R := \Re\left(\mathrm{e}^{i(\pi/2-\beta)}b(u,u)\right) \geq \sin\beta\|\nabla u\|_2^2 + \int\limits_{S_H} \sin(\arg(k^2)-\beta)|k^2|\,|u|^2dx$$

$$\geq \sin\beta\|\nabla u\|_2^2 + \sin(\theta-\beta)\frac{k_-^2}{k_+^2}k_+^2\|u\|_2 = \sin\beta\left(\|\nabla u\|_2^2 + \alpha\frac{k_-^2}{k_+^2}k_+^2\|u\|_2^2\right).$$

Thus, and by Lemma 5, for $0 \leq \gamma \leq 1$,

$$R \geq \sin\beta\left(\gamma\,\|\nabla u\|_2^2 + \frac{2(1-\gamma)+\alpha\kappa_-^2}{\kappa_+^2}\,k_+^2\|u\|_2\right).$$

Choosing first $\gamma = 1$ and $\alpha = \kappa_+^2/\kappa_-^2$, we see that

$$R \geq \sin\beta\|u\|_{V_H}^2 \geq \frac{\sin\theta}{1+\kappa_+^2/\kappa_-^2}\,\|u\|_{V_H}^2.$$

Alternatively, choosing $\gamma = 2/(2+\kappa_+^2)$ and $\alpha = 0$, so that $\beta = \theta$, we see that

$$R \geq \frac{\sin\theta}{1+\kappa_+^2/2}\,\|u\|_{V_H}^2.$$

■

Theorem 4 now follows from Lemmas 8 and 9 and the Lax-Milgram lemma. The final bound (17) is a consequence of the definition of the norm on V_H and of Lemma 5. These imply, in the particular case that $\mathcal{G}(v) := -(g,v)$ for some $g \in L^2(S_H)$, that

$$\|\mathcal{G}\|_{V_H^*} = \sup_{v\in V_H}\frac{|(v,g)|}{\|v\|_{V_H}} \leq \sup_{v\in V_H}\frac{\|v\|_2\|g\|_2}{\|v\|_{V_H}} \leq \max\left(k_+^{-1}, \frac{H-f_-}{\sqrt{2}}\right)\|g\|_2.$$

4 Ellipticity in weighted spaces

In this final section we study the variational problem in a weighted space setting. Given $w : \overline{S_H} \to \mathbb{R}$ which satisfies that $w \in C^1(\overline{S_H})$, and that $w(x) > 0$ for $x \in \overline{S_H}$, we define weighted versions of the spaces $L^2(S_H)$ and V_H by

$$L^2_w(S_H) := \{\phi \in L^2_{\mathrm{loc}}(S_H) : w\phi \in L^2(S_H)\},\ V_{H,w} := \{\phi \in H^1_{\mathrm{loc}}(S_H) : w\phi \in V_H\}.$$

Both of these spaces are Hilbert spaces when equipped with the appropriate scalar product. In particular, the induced norms are, respectively, $\|\cdot\|_{2,w}$ and $\|\cdot\|_{V_{H,w}}$, defined by

$$\|\phi\|_{2,w} := \|w\phi\|_2,\quad \|\phi\|_{V_{H,w}} := \|w\phi\|_{V_H}.$$

We let $V_{H,w}^*$ denote the space of continuous anti-linear functionals on $V_{H,w}$.

We have shown that, at least in certain cases discussed in the previous section, our boundary value problem is well-posed; for every $g \in L^2(S_H)$ there exists exactly one solution to the boundary value problem, with $v \in V_a$ for every $a > f_+$, and $\|v\|_{V_H}$ bounded in terms of $\|g\|_2$ by Theorem 4.

We will extend these results to a certain class of weighted spaces via a study of the following generalization of the variational formulation (13) of the boundary value problem: given $g \in L^2_w(S_H)$ find $u \in V_{H,w}$ such that

$$b(u,v) = -(g,v), \quad v \in V_{H,1/w}. \tag{18}$$

The weights we shall consider are those which are constant on Γ_H and satisfy, for some $M > 0$, the inequality

$$\frac{|\nabla w(x)|}{k_+ w(x)} \leq M, \quad x \in \overline{S_H}. \tag{19}$$

These conditions are satisfied, for example, if

$$w(x) := \exp(\eta x_n), \quad x \in \overline{S_H}, \tag{20}$$

for some $\eta \in \mathbb{R}$, with $M = |\eta|/k_+$.

The assumptions (19) and that w is constant on Γ_H ensure that b is a bounded sesquilinear form on $V_{H,w} \times V_{H,1/w}$. To see this note first that the operator M_w of multiplication by w is an isometric isomorphism from $L^2_w(S_H)$ to $L^2(S_H)$ and from $V_{H,w}$ to V_H. Thus $b(\cdot,\cdot)$ is bounded on $V_{H,w} \times V_{H,1/w}$ if and only if $b_w(\cdot,\cdot)$ is bounded on $V_H \times V_H$, where

$$b_w(u,v) := b(u/w, wv), \quad u,v \in V_H.$$

We calculate, for $u,v \in V_H$, noting w is constant on Γ_H, that

$$b_w(u,v) - b(u,v) = \left(\frac{1}{w}\nabla u, v\nabla w\right) - \left(u\frac{\nabla w}{w^2}, w\nabla v\right) - \left(\frac{u}{w^2}\nabla w, v\nabla w\right)$$

so that

$$\begin{aligned} |b_w(u,v) - b(u,v)| &\leq Mk_+\|\nabla u\|_2\|v\|_2 + Mk_+\|u\|_2\|\nabla v\|_2 + M^2k_+^2\|u\|_2\|v\|_2 \\ &\leq M\|u\|_{V_H}\|v\|_{V_H} + M^2k_+^2\|u\|_2\|v\|_2 \\ &\leq M(1+M)\|u\|_{V_H}\|v\|_{V_H}. \end{aligned} \tag{21}$$

This bound implies that $b_w(\cdot,\cdot) - b(\cdot,\cdot)$, and hence $b_w(\cdot,\cdot)$ itself, are bounded on $V_H \times V_H$ so that $b(\cdot,\cdot)$ is bounded on $V_{H,w} \times V_{H,1/w}$. Moreover, combined with Theorem 4, it establishes the V_H-ellipticity of $b_w(\cdot,\cdot)$ under the conditions of Theorem 4, if M is small enough.

Theorem 10 *Suppose that $w(x)$ is constant on Γ_H and satisfies (19). Then $b(\cdot,\cdot)$ is bounded on $V_{H,w} \times V_{H,1/w}$ and $b_w(\cdot,\cdot)$ is bounded on $V_H \times V_H$, with*

$$\begin{aligned}\sup_{u\in V_{H,w},\, v\in V_{H,1/w}} \frac{|b(u,v)|}{\|u\|_{V_{H,w}}\,\|v\|_{V_{H,1/w}}} &= \sup_{u\in V_H,\, v\in V_H} \frac{|b_w(u,v)|}{\|u\|_{V_H}\,\|v\|_{V_H}} \\ &\le 1 + \frac{k_\infty^2}{k_+^2} + M(1+M).\end{aligned}$$

If also $\beta := N - M(1+M) > 0$, where

$$N := \max\left(\frac{2-\kappa_\infty^2}{2+\kappa_+^2}, \frac{\sin\theta}{1+\kappa_+^2/\max(2,\kappa_-^2)}\right),$$

then

$$|b_w(u,u)| \ge \beta \|u\|_{V_H}^2, \quad u \in V_H.$$

It follows that the variational problem: given $\mathcal{G} \in V_{H,1/w}^*$, find $u \in V_{H,w}$ such that

$$b(u,v) = \mathcal{G}(v), \quad v \in V_{H,1/w},$$

is uniquely solvable and the solution satisfies the estimate

$$\|u\|_{V_{H,w}} \le \beta^{-1} \|\mathcal{G}\|_{V_{H,1/w}^*}. \tag{22}$$

In particular, the scattering problem (18) is uniquely solvable and the solution satisfies the bound

$$k_+^2 \|wu\|_2 \le k_+ \|wu\|_{V_H} = k_+ \|u\|_{V_{H,w}} \le \beta^{-1} \|g\|_{2,w} = \beta^{-1} \|wg\|_2. \tag{23}$$

We note that $\beta > 0$ if and only if $M < N/(1/2 + \sqrt{N + 1/4}\,)$, which clearly holds if

$$M < \frac{\tilde{N}}{\frac{1}{2} + \sqrt{\tilde{N} + \frac{1}{4}}},$$

where

$$\tilde{N} := \frac{\sin\theta}{1 + \kappa_+^2/\kappa_-^2}.$$

If, for some $k_i > 0$,

$$\Im k(x) \ge k_i, \quad x \in S_H, \tag{24}$$

then $\arg k(x) \geq \chi := \sin^{-1}(k_i/k_\infty)$, $x \in S_H$, so that (15) holds with $\theta = 2\chi$ and, defining $k_r = k_\infty \cos\chi$, we have that

$$\tilde{N} = \frac{2k_i k_r}{k_\infty^2(1 + k_+^2/k_-^2)}. \tag{25}$$

Consider now the case when w is given by (20), and note that, since x_n is bounded on S_H, $V_{H,w} = V_H$ so that, for this weight, a large part of the above theorem (though not the explicit bounds) follows already from Theorem 4. Thus, noting the above observations, we obtain the following corollary of the above theorem and Theorems 3 and 4.

Corollary 11 *If (24) holds for some $k_i > 0$ then the boundary value problem has at most one solution. Moreover, for*

$$|\eta| < \eta_{\max} := \frac{k_+\tilde{N}}{\frac{1}{2} + \sqrt{\tilde{N} + \frac{1}{4}}},$$

where $\tilde{N}$ is given by (25), it holds that

$$k_+^2 \left\{ \int_{S_H} \mathrm{e}^{2\eta x_n} |u(x)|^2 \, dx \right\}^{1/2} \leq \beta^{-1} \left\{ \int_{S_H} \mathrm{e}^{2\eta x_n} |g(x)|^2 \, dx \right\}^{1/2}, \tag{26}$$

with $\beta = \tilde{N} - \eta(1 + \eta/k_+)/k_+$.

We note that, in the above corollary,

$$\eta_{\max} < k_+\tilde{N} < \frac{2k_i k_+ k_\infty}{k_\infty^2 + k_+^2} < k_i \tag{27}$$

and that, in the case when k is constant in S_H with modulus k_+, so that $k_+ = k_\infty = k_-$, it holds that $\eta_{\max} \sim k_+\tilde{N} \sim k_i$ as $k_i \to 0$ with k_+ fixed, so that the bound (27) is sharp in this limit. Of course the bound (27), limiting the range of validity of (26) to $|\eta| < \eta_{\max} < k_i$, is to be expected. In the case when k is constant in S_H, a vertically travelling plane wave in S_H has the form $\exp(\pm(ik_r x_n - k_i x_n))$, so it is reasonable to suppose that $\int_{S_H} \mathrm{e}^{2\eta x_n} |u(x)|^2 \, dx$ cannot be bounded, independently of H and of the location of ∂D, for $|\eta| = k_i$ for every compactly supported g.

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