JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 15, Number 1, Spring 2003

SOLVABILITY AND SPECTRAL PROPERTIES OF INTEGRAL EQUATIONS ON THE REAL LINE: II. L^p-SPACES AND APPLICATIONS

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Dedicated to Professor Ian Sloan on the occasion of his 65th birthday.

ABSTRACT. We consider the solvability of linear integral equations on the real line, in operator form $(\lambda - K)\phi = \psi$, where $\lambda \in \mathbf{C}$ and K is an integral operator. We impose conditions on the kernel, k, of K which ensure that K is bounded as an operator on $L^p(\mathbf{R})$, $1 \leq p \leq \infty$, and on $BC(\mathbf{R})$. We establish conditions on families of operators, $\{K_k : k \in W\}$, which ensure that if $\lambda \neq 0$ and $\lambda\phi = K_k\phi$ has only the trivial solution in $BC(\mathbf{R})$, for all $k \in W$, then for $1 \leq p \leq \infty$, $(\lambda - K)\phi = \psi$ has exactly one solution $\phi \in L^p(\mathbf{R})$ for every $k \in W$ and $\psi \in L^p(\mathbf{R})$. The results of considerable generality apply in particular to kernels of the form $k(s,t) = \kappa(s-t)z(t)$ and $k(s,t) = \tilde{\kappa}(s-t)\tilde{z}(s,t)$, where $\kappa, \tilde{\kappa} \in L^1(\mathbf{R})$, $z \in L^\infty(\mathbf{R})$, $\tilde{z} \in BC(\mathbf{R}^2)$ and $\tilde{\kappa}(s) = O(s^{-b})$ as $|s| \to \infty$, for some $b > 1$. As a significant application we consider the problem of acoustic scattering by a sound-soft, unbounded one-dimensional rough surface which we reformulate as a second kind boundary integral equation. Combining the general results of earlier sections with a uniqueness result for the boundary value problem, we establish that the integral equation is well-posed as an equation on $L^p(\mathbf{R})$, $1 \leq p \leq \infty$, and on weighted spaces of continuous functions.

1. Introduction. We consider in this paper integral equations of the form

$$\lambda\phi(s) - \int_{-\infty}^{+\infty} k(s,t)\phi(t)\,dt = \psi(s), \quad s \in \mathbf{R}, \tag{1.1}$$

where $\lambda \in \mathbf{C}$, the functions $k : \mathbf{R}^2 \to \mathbf{C}$ and ψ are assumed known and ϕ is the solution to be determined. Define the integral operator K by

$$K\psi(s) = \int_{-\infty}^{+\infty} k(s,t)\psi(t)\,dt, \quad s \in \mathbf{R}. \tag{1.2}$$

Received by the editors on September 25, 2002.

Then (1.1) can be abbreviated in operator notation as

$$(\lambda - K)\phi = \psi. \tag{1.3}$$

We assume throughout that k is (Lebesgue) measurable and that the following assumptions on k hold:

A. $\displaystyle\sup_{s\in\mathbf{R}} \int_{-\infty}^{+\infty} |k(s,t)|\,dt < \infty.$

B. For all $s \in \mathbf{R}$, $\displaystyle\int_{-\infty}^{+\infty} |k(s,t) - k(s',t)|\,dt \longrightarrow 0,$ as $s' \to s$.

Assumption A ensures that K is a bounded operator on $L^\infty(\mathbf{R})$. (As usual, $L^p(\mathbf{R})$, $1 \le p \le \infty$, denotes the Banach space of measurable functions $\phi : \mathbf{R} \to \mathbf{C}$ for which $|\phi|^p$ is integrable, $1 \le p < \infty$, ϕ is essentially bounded, $p = \infty$.) Assumptions A and B together ensure that $K : L^\infty(\mathbf{R}) \to BC(\mathbf{R})$ and is bounded, where $BC(\mathbf{R}) \subset L^\infty(\mathbf{R})$ is the subspace of functions bounded and continuous on $\mathbf{R}$.

For much of the paper we will make the following stronger assumption than A:

A′. For some $\kappa \in L^1(\mathbf{R})$,

$$|k(s,t)| \le |\kappa(s-t)|, \quad s,t \in \mathbf{R}.$$

Assumption A' implies that, by *Young's equality*, $K : L^p(\mathbf{R}) \to L^p(\mathbf{R})$. and is bounded, for $1 \le p \le \infty$, with norm $\|K\| \le \|\kappa\|_1$.

A main concern of this paper is to study the solvability of (1.1) as an operator equation on $L^p(\mathbf{R})$, $1 \le p \le \infty$, and other function spaces. Let Y denote any one of these function spaces. Then a main aim of the paper is to establish conditions, sufficiently explicit that they can be checked in applications, which ensure that (1.1) has exactly one solution $\phi \in Y$ for every $\psi \in Y$. In terms of our operator notation, the aim is thus to seek conditions which ensure that $(\lambda - K) : Y \to Y$ is bijective, in which case $(\lambda - K)^{-1} : Y \to Y$ exists and, by the Banach theorem, is bounded. Thus the paper contains results giving conditions which ensure, for specific function spaces Y, that $(\lambda - K)^{-1} \in B(Y)$, the set of bounded linear operators on Y.

Given a Banach space Y and $A \in B(Y)$, let $\|A\|_Y$ denote the norm of $A : Y \to Y$, $\mathcal{R}_Y(A) := \{\lambda \in \mathbf{C} : (\lambda - A)^{-1} \in B(Y)\}$, and let $\Sigma_Y(A) := \mathbf{C} \setminus \mathcal{R}_Y(A)$ denote the spectrum of $A \in B(Y)$. Let $\Sigma_Y^p(A) \subset \Sigma_Y(A)$ denote the point spectrum, the set of eigenvalues of A, in other words the set of λ for which $(\lambda - A) : Y \to Y$ is not injective.

In terms of these definitions and supposing that $K \in B(Y)$, clearly (1.3) has exactly one solution $\phi \in Y$ for every $\psi \in Y$ if and only if $\lambda \notin \Sigma_Y(K)$. A second aim of the paper, clearly related to the first aim mentioned above, is to shed light on the relationships between $\Sigma_Y(K)$ and $\Sigma_Y^p(K)$ and between $\Sigma_Y(K)$ and $\Sigma_Z(K)$ for various function spaces Y and Z.

For the special case satisfying A' and B, of the convolution integral equation on the real line, when $k(s,t) = \kappa(s-t)$, $s,t \in \mathbf{R}$, with $\kappa \in L^1(\mathbf{R})$, the questions considered in this paper are already well understood. For brevity, in the remainder of the paper, let us abbreviate $L^p(\mathbf{R})$ by L^p and $BC(\mathbf{R})$ by X. Then, in particular [**24**],

$$\Sigma_{L^p}(K) = \Sigma_X(K) = \{0\} \cup \Sigma_X^p(K) = \{0\} \cup \{\hat{\kappa}(\xi) : \xi \in \mathbf{R}\}, \tag{1.4}$$

for $1 \le p \le \infty$, where $\hat{\kappa}(\xi) := \int_{-\infty}^{+\infty} \kappa(t) e^{i\xi t}\, dt$, $\xi \in \mathbf{R}$, is the Fourier transform of κ. A large part of the argument in this paper can be viewed as an attempt to investigate in what way the equation

$$\Sigma_{L^p}(K) = \Sigma_X(K) = \{0\} \cup \Sigma_X^p(K)$$

generalizes to cases where the kernel $k(s,t)$, while not an L^1 convolution kernel, is bounded by such a kernel, satisfying A'.

In Section 2 we review results drawn from Part I of this study [**6**] which are essential for the arguments we make in later sections. The earlier paper considers the solvability of (1.1) in $BC(\mathbf{R})$ and in weighted spaces of continuous functions, defined for $a \ge 0$ by

$$X_a := \{\phi \in X : \phi(s) = O(s^{-a}), |s| \to \infty\}. \tag{1.5}$$

X_a is a Banach space under the norm $\|\cdot\|_{X_a}$, defined by $\|\phi\|_{X_a} = \|\phi w_a\|_\infty$, where $w_a(s) = (1+|s|)^a$. It is shown in [**9**] that, if k satisfies A' and B and

$$\kappa(s) = O(s^{-b}), \quad |s| \to \infty, \tag{1.6}$$

for some $b > 1$, then $K \in B(X_a)$ for $0 \le a \le b$. A main result of [**6**] is that these same assumptions ensure that

$$\Sigma_{X_a}(K) = \Sigma_X(K), \quad 0 \le a \le b. \tag{1.7}$$

In Section 2 we also review results in [**6**] on the relationship between $\Sigma_X(K)$ and $\Sigma_X^p(K)$. It is shown in [**15**] that, if for some $r \in \mathbf{R} \setminus \{0\}$,

$$k(s+r, t+r) = k(s,t), \quad s,t \in \mathbf{R}, \tag{1.8}$$

and A and B hold, then

$$\Sigma_X(K) = \Sigma_X^p(K) \cup \{0\}. \tag{1.9}$$

It is easy to see that (1.8) is equivalent to the operator equation

$$T_r K = K T_r, \tag{1.10}$$

where T_r is the translation operator defined by

$$T_r \psi(s) = \psi(s-r), \quad s \in \mathbf{R}. \tag{1.11}$$

Obviously (1.8) and (1.10) hold for all $r \in \mathbf{R}$ in the case when $k(s,t) = \kappa(s-t)$.

But (1.9) certainly does not hold for all kernels satisfying A and B. As an example, (1.9) does not hold in general for the Wiener-Hopf cases,

$$k(s,t) = \begin{cases} \kappa(s-t) & s \in \mathbf{R},\ t \geq 0, \\ 0 & s \in \mathbf{R},\ t < 0, \end{cases} \tag{1.12}$$

with $\kappa \in L^1$ [**24, 27**].

However, the following result, which may be viewed as a generalization of (1.9) is shown in [**18**]. Denote the integral operator K by K_k to indicate its dependence on its kernel k, and let W denote a family of kernels which satisfy A and B uniformly and have the translation invariance property, cf. (1.10), that

$$\{T_r K_k : k \in W\} = \{K_k T_r : k \in W\}, \tag{1.13}$$

for some $r \in \mathbf{R} \setminus \{0\}$. Then, provided W also has certain compactness properties (explained in Section 2, see Theorem 2.1 below) with respect to the σ-topology proposed in [**18**], it holds that

$$\bigcup_{k \in W} \Sigma_X(K_k) = \{0\} \cup \bigcup_{k \in W} \Sigma_X^p(K_k) =: \Sigma_X^p(W).$$

Moreover, for $\lambda \notin \Sigma_X^p(W)$, it holds that

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\|_X < \infty.$$

We note that the idea of considering a family of operators satisfying A and B uniformly and having the property (1.13) derives from an analysis of the finite section method for the Wiener-Hopf cases (1.12) in Anselone and Sloan [**1, 2**].

In Section 3 we obtain solvability results in L^p-spaces via the solvability results in $BC(\mathbf{R})$ and in weighted spaces of [**6**], summarized in Section 2, and via a consideration of properties of the adjoint operator K^T with transposed kernel k^T, defined by

$$k^T(s,t) := k(t,s), \quad s,t \in \mathbf{R}.$$

Generalizing (1.12), in Section 3.1 we consider kernels of the form

$$k(s,t) = \kappa(s-t)z(t), \tag{1.14}$$

for some $\kappa \in L^1$, $z \in L^\infty$, so that $K = \mathcal{K}M_z$ where $\mathcal{K}$ is the convolution integral operator with kernel $\kappa(s-t)$ and M_z is the operation of multiplication by z. In the case that κ is even, we show that, cf. (1.4),

$$\Sigma_{L^p}(\mathcal{K}M_z) \subset \Sigma_{L^1}(\mathcal{K}M_z) = \Sigma_{L^\infty}(\mathcal{K}M_z) = \Sigma_X(\mathcal{K}M_z),$$

$1 < p < \infty$. In the case that $V \subset L^\infty$ is weak$*$-sequentially compact and has certain translation and reflection invariance properties, specified in Corollary 3.11 below, we show that

$$\bigcup_{z \in V} \Sigma_{L^p}(\mathcal{K}M_z) \subseteq \{0\} \cup \bigcup_{z \in V} \Sigma_X^p(\mathcal{K}M_z), \quad 1 \le p \le \infty, \tag{1.15}$$

with equality in (1.15) for $p = 1, \infty$. As an interesting example which arises in an application to an elliptic boundary value problem, (1.15) holds for the case when $V = \{z \in L^\infty : \text{ess.range}\, z \subset Q\}$ with $Q \subset \mathbf{C}$ compact and convex.

In Section 3.2 we consider a more general class of kernels but with conditions which exclude the case (1.14) if $z \notin X$. We suppose that W has the translation invariance property (1.13), for some $r \in \mathbf{R} \setminus \{0\}$, and that W and $W^T := \{k^T : k \in W\}$ satisfy A' and B uniformly with respect to k and are sequentially compact with respect to the σ-topology introduced in Section 2. Our main results are similar to (1.15), but obtained by substantially different arguments, in particular using additional results from Section 2. We show in this case, provided (1.6) holds for some $b > 1$, that

$$\bigcup_{k\in W} \Sigma_{L^p}(K_k) \subseteq \{0\} \cup \bigcup_{k\in W} \Sigma_X^p(K_k). \tag{1.16}$$

Similarly to (1.15), (1.16) is shown to hold with equality for $p = 1, \infty$. We also point out that $\Sigma_{L^p}(K_k) = \Sigma_{L^q}(K_{k^T})$, $1 \le p \le \infty$, with $1/p + 1/q = 1$. Thus (1.16) also relates the spectrum of $K_k^T = K_{k^T}$ in L^p to the point spectrum of K_k in X.

To finish Section 3.2 we present a particularly simple form of the theory, with substantially simpler conditions to verify, for the special case when the kernel takes the form

$$k(s,t) = \kappa(s-t)z(s,t), \tag{1.17}$$

for some $\kappa \in L^1$, $z \in BC(\tilde{\mathbf{R}}^2)$ where $\tilde{\mathbf{R}}^2 := \mathbf{R}^2 \setminus \{(s,s) : s \in \mathbf{R}\}$.

Section 4 considers a substantial application of the results of the previous sections. The problem considered is one of scattering by a one-dimensional unbounded rough surface. Precisely, we consider the Dirichlet boundary value problem for the Helmholtz equation in a nonlocally perturbed half-plane. The theoretical basis for boundary integral equation methods for the Helmholtz equation for scattering by unbounded surfaces is in its infancy. But recently [**13**], a novel boundary integral equation formulation for this problem has been proposed and this formulation has been shown to be uniquely solvable in the space of bounded continuous functions, leading to the first proof of unique existence of solution for the problem of scattering of a plane wave by a one-dimensional rough surface [**14**]. In this paper, applying the results of Sections 2 and 3, we show that the boundary integral equation is well posed also in L^p-spaces and weighted spaces of continuous functions. We show similar results for an alternative boundary integral equation formulation, whose integral operator is the adjoint of that in the first equation considered. This alternative integral equation formulation is obtained by a direct approach using Green's theorem, in contrast to the indirect formulation of [**13**].

Arens [**3, 4, 5**], in related work to that of Section 4, considers the problem of elastic wave scattering by one-dimensional rough surfaces, proposing a novel reformulation as a coupled pair of boundary integral equations. Employing the results of [**22, 23**] and arguments closely related to those in Section 3.2, the well posedness of the integral equation system in L^p-spaces is established [**4, 5**], leading to an existence proof for the elastic wave boundary value problem.

We finish this introduction by contrasting the results of this paper to those of earlier authors. There exists a large literature (e.g., [**7, 8, 21, 26**] and the references therein) concerned with extensions of the case $k(s,t) = \kappa(s-t)$, $k \in L^1$, and the Wiener-Hopf case (1.12) to more general kernels satisfying A' (and to systems of equations, multi-dimensional cases and singular integral operators). In large part the concern in this literature has been to characterize explicitly the spectrum or essential spectrum of the operator and give an explicit formula for the index when the operator is Fredholm. The results available provide information relating to cases we consider in this paper, in particular, regarding kernels of the form (1.14) or (1.17), with $z \in L^\infty(\mathbf{R})$ or $z \in L^\infty(\mathbf{R}^2)$, respectively. However, the results which have been obtained only apply to cases where the behavior of z is severely constrained at $\pm\infty$.

As an indication of the results that have been shown, consider the case when the kernel is given by (1.17) with $z \in L^\infty(\mathbf{R}^2)$ and suppose that z has limits at infinity, $z_\pm$, in the sense that

(1.18)

$$\lim_{A\to\infty} \operatorname*{ess.\,sup}_{s>A,t>A} |z(s,t) - z_+| = 0, \qquad \lim_{A\to\infty} \operatorname*{ess.\,sup}_{s<-A,t<-A} |z(s,t) - z_-| = 0.$$

Then [**25, 26**]

$$\begin{aligned}\Sigma_{L^p}(K) = \{0\} &\cup \{z_\pm \hat{\kappa}(\xi) : \xi \in \mathbf{R}\} \\ &\cup \left\{\lambda : \left[\arg\left(\frac{\lambda - z_-\hat{\kappa}(\xi)}{\lambda - z_+\hat{\kappa}(\xi)}\right)\right]_{\xi=-\infty}^{+\infty} \neq 0\right\} \\ &\cup \Sigma^p_{L^p}(K), \quad 1 \le p \le \infty.\end{aligned}$$

This criterion for the solvability of (1.1) in L^p is at least as easy to check in applications as (1.16) and is arguably more explicit. But the constraint that z has limits at infinity is a severe one. For example, in the application in Section 4, z has limits in the sense (1.18) if and only if the graph of the unbounded surface approaches horizontal asymptotes at $\pm\infty$.

2. Solvability in $BC(\mathbf{R})$ and in weighted spaces of continuous functions. We begin by reviewing properties of integral operators on the real line and results of Part I of this study [**6**] that we will need for our arguments. We are concerned in this section with properties of the integral operator K as an operator on $X = BC(\mathbf{R})$ and on the weighted spaces $X_a \subset X$ introduced above.

Conditions A and B ensure that the integral operator $K \in B(X)$ with

$$\|K\|_X = \sup_{s\in\mathbf{R}} \int_{-\infty}^{+\infty} |k(s,t)|\, dt.$$

Conditions A and B do not imply that K is compact. (For example, if $k(s,t) = \kappa(s-t)$, $s,t \in \mathbf{R}$, with $\kappa \in L^1$, then k satisfies A and B, but K has the continuous spectrum (1.4), so is not compact.) In Sections 3 and 4 we will use the fact that K is compact if k satisfies A, B and the following assumption [**1**].

C.

$$\int_{-\infty}^{+\infty} |k(s,t)|\, dt \longrightarrow 0 \quad \text{as } |s| \longrightarrow \infty.$$

In the case that (1.8) holds for some $r \in \mathbf{R} \setminus \{0\}$, i.e.,

$$T_r K = K T_r, \tag{2.1}$$

where T_r is the translation operator given by (1.11), we have mentioned already in the introduction that

$$\Sigma_X(K) = \Sigma_X^p(K) \cup \{0\}. \tag{2.2}$$

We have also pointed out, with an example, that (2.2) does not hold in general given only that k satisfies A and B. However, a version of (2.2) holds for families of integral operators satisfying A and B uniformly and having a translation invariance property to replace (2.1).

Let $\mathcal{K} := \{k : \mathbf{R}^2 \to \mathbf{C} : k \text{ is measurable and satisfies } A \text{ and } B\}$. For $k \in \mathcal{K}$ let K_k denote the integral operator defined by

$$K_k\phi(s) = \int_{-\infty}^{+\infty} k(s,t)\phi(t)\, dt, \quad s \in \mathbf{R}.$$

We consider families $W \subset \mathcal{K}$ satisfying the following uniform versions of A and B.

A_u.

$$\sup_{k \in W} |||k||| < \infty,$$

where

$$|||k||| := \sup_{s \in \mathbf{R}} \int_{-\infty}^{+\infty} |k(s,t)|\, dt.$$

B_u. For all $s \in \mathbf{R}$,

$$\sup_{k \in W} \int_{-\infty}^{+\infty} |k(s,t) - k(s',t)|\, dt \longrightarrow 0$$

as $s' \to s$.

For $(k_n) \subset \mathcal{K}$, $k \in \mathcal{K}$, we will write $k_n \xrightarrow{\sigma} k$ if $\sup_n |||k_n||| < \infty$ and, for all $\psi \in L^\infty$,

$$\int_{-\infty}^{+\infty} k_n(s,t)\psi(t)\, dt \longrightarrow \int_{-\infty}^{+\infty} k(s,t)\psi(t)\, dt,$$

uniformly on every finite interval. (This is convergence in the σ-topology of [**18**].) Call $W \subset \mathcal{K}$ σ-sequentially compact if every sequence $(k_n) \subset W$ has a subsequence which is σ-convergent to some $k \in W$. Clearly, if $W \in \mathcal{K}$ is σ-sequentially compact, then W satisfies A_u.

Let the translation operator $T_r^{(2)}$ be defined for $r \in \mathbf{R}$ by

$$T_r^{(2)} k(s,t) = k(s-r, t-r), \quad s,t \in \mathbf{R}.$$

The following results are shown in [**18**] and [**6**].

Theorem 2.1 [**18**, Theorem 2.10]. *Suppose that* $\lambda \neq 0$, $W \subset \mathcal{K}$ *and*

(i) W *satisfies* B_u,

(ii) W *is* σ*-sequentially compact,*

(iii) $T_r^{(2)}(W) = W$ *for some* $r \in \mathbf{R} \setminus \{0\}$ *so that*

$$\{T_r K_k : k \in W\} = \{K_k T_r : k \in W\};$$

(iv) $\lambda \notin \Sigma^p_X(K_k)$, $k \in W$.

Then $(\lambda - K_k)(X)$ *is closed for all* $k \in W$ *so that* $(\lambda - K_k)^{-1} : (\lambda - K)(X) \to X$ *is bounded. Moreover, these inverse operators are uniformly bounded, i.e.,*

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\|_X < \infty. \tag{2.3}$$

Suppose that, in addition to (i)–(iv)*, it holds that*:

(v) *For every* $\tilde{k} \in W$ *there exists* $(k_n) \subset W$ *such that* $k_n \xrightarrow{\sigma} \tilde{k}$ *and*

$$\lambda \notin \bigcup_{k \in W} \Sigma^p_X(K_k) \Longrightarrow \lambda \notin \Sigma_X(K_{k_n}) \tag{2.4}$$

for each n.

Then $(\lambda - K_k)(X) = X$ *for all* $k \in W$ *so that* $(\lambda - K_k)^{-1} \in B(X)$, $k \in W$.

We point out that (2.4) certainly holds if $\lambda - K_{k_n}$ is Fredholm of index zero. In particular, for any $\lambda \neq 0$, (2.4) holds if K_{k_n} is compact.

The next theorem requires that A' is satisfied uniformly for $k \in W$, i.e., that the following uniform version of A' holds.

A'_u. For some $\kappa \in L^1$,

$$|k(s,t)| \leq |\kappa(s-t)|, \quad s, t \in \mathbf{R},$$

for all $k \in W$.

Theorem 2.2 [**6**, Theorem 3.6]. *Suppose that* W *satisfies* A'_u *and* B_u*, that* (1.6) *holds for some* $b > 1$ *and that* W *is* σ*-sequentially compact. Then*

$$\Sigma_{X_a}(K_k) = \Sigma_X(K_k), \quad k \in W,\ 0 \leq a \leq b, \tag{2.5}$$

and, for $\lambda \notin \cup_{k \in W} \Sigma_X(K_k)$*, it holds that*

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\|_{X_a} < \infty \Longleftrightarrow \sup_{k \in W} \|(\lambda - K_k)^{-1}\|_X < \infty,$$

$0 \le a \le b$.

Combining Theorems 2.1 and 2.2, we have the following criterion for $(\lambda - K_k)^{-1} \in B(X_a)$.

Theorem 2.3. *Suppose that* $\lambda \neq 0$ *and*

(i) $W \subset \mathcal{K}$ *satisfies* A'_u *and* B_u *and* (1.6) *holds with* $b > 1$,

(ii) W *is* σ*-sequentially compact*,

(iii) $T_r^{(2)}(W) = W$ *for some* $r \in \mathbf{R} \setminus \{0\}$,

(iv) $\lambda \notin \Sigma_X^p(K_k)$, $k \in W$,

(v) *For every* $\tilde{k} \in W$ *there exists* $(k_n) \subset W$ *such that* $k_n \xrightarrow{\sigma} \tilde{k}$ *and*

$$\lambda \notin \bigcup_{k \in W} \Sigma_X^p(K_k) \Longrightarrow \lambda \notin \Sigma_X(K_{k_n})$$

for each n.

Then $\lambda \notin \Sigma_{X_a}(K_k)$ *for* $k \in W$, $0 \le a \le b$ *and*

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\|_{X_a} < \infty, \quad 0 \le a \le b.$$

3. Solvability in L^p-spaces. The results of the previous section examine the solvability of (1.1) in the weighted space X_a. In this section we apply these results to examine the solvability of (1.1) and its adjoint equation

$$\lambda\phi(s) - \int_{-\infty}^{+\infty} k(t,s)\phi(t)\,dt = \psi(s), \quad s \in \mathbf{R}, \tag{3.1}$$

in X and in L^p, $1 \le p \le \infty$. Besides the work of Section 2, our main tool will be results relating properties of an operator $A \in B(Y)$ to properties of its adjoint $A^* \in B(Y^*)$, where Y^* denotes the dual space of the Banach space Y. In particular, the following standard results will suffice for the arguments which follow, see, e.g., [**28**].

Theorem 3.1. *Suppose that* Y *is a Banach space,* Y^* *is its dual space,* $A \in B(Y)$ *and* $A^* \in B(Y^*)$ *is the adjoint of* A. *Then*

(i) $\|A\| = \|A^*\|$.

(ii) $A(Y)$ *is dense in* Y *if and only if* A^* *is injective.*

(iii) A *is bijective if and only if* A^* *is bijective and, if they are both bijective, then* $(A^*)^{-1} = (A^{-1})^*$*, the adjoint of* A^{-1} *so that* $\|A^{-1}\| = \|(A^*)^{-1}\|$.

Of course, these results are useful to us since, for $1 \leq p < \infty$, $(L^p)^*$ can be identified with L^q where q here and in the remainder of the paper is related to p via the equation $1/p + 1/q = 1$ (with $q = \infty$ if $p = 1$, $q = 1$ if $p = \infty$). This identification can be made via the isometric isomorphism $i : L^q \to (L^p)^*$ given by $i(\phi) = \tilde{\phi}$ where $\tilde{\phi}$ is defined by

$$\tilde{\phi}(\psi) = \int_{-\infty}^{+\infty} \phi(s)\psi(s)\, ds, \quad \psi \in L^p.$$

Making this identification, it follows from Fubini's theorem (see, e.g., Jörgens [**24**]) that if the integral operator K given by (1.2) satisfies $K \in B(L^p)$ for some $p \in [1, \infty)$, its adjoint operator $K^* \in B((L^p)^*) = B(L^q)$ is the integral operator K^T given by (1.2) with K, k replaced by K^T, k^T where $k^T(s,t) := k(t,s)$, $s, t \in \mathbf{R}$. That is,

$$K^T\phi(s) = \int_{-\infty}^{+\infty} k(t,s)\phi(t)\, dt, \quad s \in \mathbf{R}. \tag{3.2}$$

We shall consider cases when both k and k^T satisfy Assumption A. Then the following result holds [**24**, Chapter 11].

Theorem 3.2. *If* k *and* k^T *satisfy* A*, then* K *and* K^T*, defined by* (1.2) *and* (3.2)*, respectively, are bounded operators on* L^p*,* $1 \leq p \leq \infty$*. Further, for all* $\lambda \in \mathbf{C}$ *and* $1 \leq p < \infty$*,* $(\lambda - K^T) \in B(L^q) = B((L^p)^*)$ *is the adjoint of* $(\lambda - K) \in B(L^p)$ *and* $(\lambda - K) \in B(L^q)$ *is the adjoint of* $(\lambda - K^T) \in B(L^p)$*.*

In part the above theorem can be established using the following result. This is a special case of the interpolation theorem of Riesz-Thorin, often called the Riesz convexity theorem [**29**, Chapter V, Theorem 1.3].

Theorem 3.3. *If, for some $s, r \in [1, \infty]$, $A \in B(L^r) \cap B(L^s)$, then for*

$$\frac{1}{p} = \frac{t}{r} + \frac{1-t}{s},$$

with $0 < t < 1$, it holds that $A \in B(L^p)$ with

$$\|A\|_{L^p} \leq \|A\|_{L^r}^t \|A\|_{L^s}^{1-t}. \tag{3.3}$$

Clearly, as a simple consequence of Theorems 3.1 and 3.2 we have that, if k and k^T satisfy A, then

$$\Sigma_{L^p}(K) = \Sigma_{L^q}(K^T), \quad 1 \leq p \leq \infty. \tag{3.4}$$

We can also, if k additionally satisfies B, relate $\Sigma_{L^\infty}(K)$ to $\Sigma_X(K)$ via the following simple result.

Lemma 3.4. *Suppose Y is a Banach space, $Z \subset Y$ is a closed subspace and $A \in B(Y)$ with $A(Y) \subset Z$. Then, for $\lambda \neq 0$, $(\lambda - A)^{-1} \in B(Y)$ if and only if $(\lambda - A)^{-1} \in B(Z)$ and*

$$\|(\lambda - A)^{-1}\|_Z \leq \|(\lambda - A)^{-1}\|_Y \leq |\lambda|^{-1}(1 + \|(\lambda - A)^{-1}\|_Z \|A\|_Y). \tag{3.5}$$

Proof. Suppose that $\lambda \neq 0$. It is clear, since $A(Y) \subset Z$, that $\lambda - A$ has the same kernel in Z as in Y. Further, since $A(Y) \subset Z$, it is easy to see that $(\lambda - A)(Y) = Y$ implies $(\lambda - A)(Z) = Z$. Conversely, if $(\lambda - A)(Z) = Z$, then, given $\psi \in Y$, there exists $\phi \in Z$ with $(\lambda - A)\phi = A\psi$ and

$$(\lambda - A)(\psi + \phi) = \lambda\psi. \tag{3.6}$$

Thus $(\lambda - A)(Y) = Y$. We have shown that $\lambda - A : Y \to Y$ is bijective if and only if $\lambda - A : Z \to Z$ is bijective. The bound (3.5) follows easily from the definitions of the norms in $B(Y)$ and $B(Z)$ and equation (3.6). □

Applying Lemma 3.4 and with some additional agreement we obtain the following result. In the proof of this lemma, we make use of weak∗-convergence in $L^\infty = (L^1)^*$. This standard topology on L^∞ is used more extensively in Section 3.1, where a definition and characterization of weak$*$-convergence in L^∞ will be given.

Lemma 3.5. *If k satisfies A and B, then $\Sigma_{L^\infty}(K) = \Sigma_X(K)$ and, if $\lambda \notin \Sigma_X(K)$, then*

$$\|(\lambda - K)^{-1}\|_{L^\infty} = \|(\lambda - K)^{-1}\|_X.$$

Proof. If k satisfies A and B then, by [**6**, Lemma 2.5], $0 \in \Sigma_X(K)$. Exactly the same proof applies to show that $0 \in \Sigma_{L^\infty}(K)$. Combining this with Lemma 3.4 we see that $\Sigma_{L^\infty}(K) = \Sigma_X(K)$. If $\lambda \notin \Sigma_X(K)$, then $\lambda \neq 0$ and $\|(\lambda - K)^{-1}\|_X \leq \|(\lambda - K)^{-1}\|_{L^\infty}$ by Lemma 3.4. To see that the reverse inequality holds, note that given $\psi \in L^\infty$ we can construct $(\psi_n) \subset X$ such that (ψ_n) converges weak$*$ to ψ in $L^\infty = (L^1)^*$ and such that $\|\psi_n\|_\infty \to \|\psi\|_\infty$. (For example, choose a compactly supported $\kappa \in C(\mathbf{R})$ with $\kappa \geq 0$ and $\int_{-\infty}^{+\infty} \kappa(t)\, dt = 1$ and define $\psi_n(s) = n \int_{-\infty}^{+\infty} \kappa(n(s-t))\psi(t)\, dt$.) Let $\phi_n = (\lambda - K)^{-1}\psi_n$. Then $\lambda\phi_n - K\phi_n = \psi_n \overset{w*}{\to} \psi$. Since (ϕ_n) is bounded, $(K\phi_n)$ is bounded and so, by the Banach-Alaoglu theorem, has a weak$*$ convergent subsequence. Denoting this subsequence by itself, we see that, for some $\phi \in L^\infty$, $\phi_n = \lambda^{-1}(K\phi_n + \psi_n) \overset{w*}{\to} \phi$. Now, since k satisfies A, $\phi_n \overset{w*}{\to} \phi \Rightarrow K\phi_n \overset{w*}{\to} K\phi$, see, e.g., [**15**]. Thus $\lambda^{-1}(K\phi_n + \psi_n) \overset{w*}{\to} \lambda^{-1}(K\phi + \psi)$ so that $\phi = (\lambda - K)^{-1}\psi$. But also

$$\begin{aligned} \|\phi\|_\infty \leq \limsup \|\phi_n\|_\infty &\leq \|(\lambda - K)^{-1}\|_X \limsup \|\psi_n\|_\infty \\ &= \|(\lambda - K)^{-1}\|_X \|\psi\|_\infty. \end{aligned}$$

Thus $\|(\lambda - K)^{-1}\|_{L^\infty} \leq \|(\lambda - K)^{-1}\|_X$. □

3.1 When K is the product of a convolution and multiplication. As a first illustration of the power of these results and of Theorem 2.1 in the last section, we consider their application to the integral equation

$$\lambda\phi(s) - \int_{-\infty}^{+\infty} \kappa(s-t)z(t)\phi(t)\, dt = \psi(s), \quad s \in \mathbf{R}, \tag{3.7}$$

and its adjoint equation

$$\lambda\phi(s) - \int_{-\infty}^{+\infty} \kappa(t-s)z(s)\phi(t)\,dt = \psi(s), \quad s \in \mathbf{R}, \tag{3.8}$$

with $\kappa \in L^1$, $z \in L^\infty$. Of course, these are equations (1.1) and (3.1), respectively, with

$$k(s,t) = \kappa(s-t)z(t), \quad k^T(s,t) = k(t,s) = \kappa(t-s)z(s).$$

In operator notation we can abbreviate (3.7) and (3.8) as $\lambda\phi - K\phi = \psi$ and $\lambda\phi - K^T\phi = \psi$, respectively, with

$$K = \mathcal{K}M_z, \quad K^T = M_z\mathcal{K}^T,$$

where M_z is the operation of multiplication by z and $\mathcal{K}$ is the convolution integral operator defined by

$$\mathcal{K}\phi(s) = \int_{-\infty}^{+\infty} \kappa(s-t)\phi(t)\,dt, \quad s \in \mathbf{R}.$$

It is easy to see that

$$\mathcal{K}^T = R\mathcal{K}R$$

where R is the reflection operator defined by

$$R\phi(s) = \phi(-s), \quad s \in \mathbf{R}.$$

Thus

$$K^T = M_zR\mathcal{K}R = RM_{Rz}\mathcal{K}R.$$

But our first result will consider the case when κ is even so that $\mathcal{K}^T = \mathcal{K}$. In addition to Theorems 3.1–3.3, it requires, for its proof, only the following simple result.

Lemma 3.6. *Suppose that, for some Banach space Y and $\lambda \neq 0$, $A, B, (\lambda - AB)^{-1} \in B(Y)$. Then also $(\lambda - BA)^{-1} \in B(Y)$ and*

$$(\lambda - BA)^{-1} = \lambda^{-1} + \lambda^{-1}B(\lambda - AB)^{-1}A. \tag{3.9}$$

Proof. We easily check that the right-hand side of (3.9) is indeed a right and left inverse for $\lambda - BA$. □

Theorem 3.7. *Suppose that* $1 \le q \le r \le p \le \infty$ (*with* $1/p+1/q=1$) *and that* $\kappa(s) = \kappa(-s)$, $s \in \mathbf{R}$. *Then the spectra of* $K = \mathcal{K}M_z$ *and* $K^T = M_z\mathcal{K}^T = M_z\mathcal{K}$ *are related by*

$$0 \in \Sigma_{L^r}(K) = \Sigma_{L^r}(K^T) \subset \Sigma_{L^p}(K) = \Sigma_{L^q}(K).$$

Further, $\Sigma_{L^\infty}(K) = \Sigma_X(K)$.

Proof. We show first that $0 \in \Sigma_{L^r}(K) \cap \Sigma_{L^r}(K^T)$, $1 \le r \le \infty$. Let $\chi \in C_0^\infty(\mathbf{R}) := \{\phi \in C^\infty(\mathbf{R}) : \phi \text{ is compactly supported}\}$ with $\chi \not\equiv 0$ and define $\phi_n(s) = \chi(s)e^{ins}$, $s \in \mathbf{R}$, $n \in \mathbf{N}$. Then $(\phi_n) \subset L^r$ with $\|\phi_n\|_r = \|\chi\|_r \neq 0$ but $\|\mathcal{K}\phi_n\|_r \to 0$ as $n \to \infty$, so that $\mathcal{K}(L^r)$ is not closed. (That $\|\mathcal{K}\phi_n\|_r \to 0$ is easy to see when $\kappa \in C_0^\infty(\mathbf{R})$ and follows from the denseness of $C_0^\infty(\mathbf{R})$ in L^1 in the general case.) Thus $\mathcal{K}$ and hence K is not surjective so that $0 \in \Sigma_{L^r}(K)$. Further, K^T is not surjective: this is clearly true if M_z is not surjective; and if M_z is surjective, then $ess. \inf_{s\in\mathbf{R}} |z(s)| > 0$ so that M_z is injective and $\mathcal{K}(L^r) \neq L^r \Rightarrow (M_z\mathcal{K})(L^r) \neq L^r$. Thus $0 \in \Sigma_{L^r}(K^T)$.

Given that $0 \in \Sigma_{L^r}(K) \cap \Sigma_{L^r}(K^T)$ it follows from Lemma 3.6 that $\Sigma_{L^r}(K) = \Sigma_{L^r}(K^T)$, $1 \le r \le \infty$. Thus, and by (3.4),

$$\Sigma_{L^p}(K) = \Sigma_{L^q}(K^T) = \Sigma_{L^q}(K).$$

It follows from Theorem 3.3 that $\Sigma_{L^r}(K) \subset \Sigma_{L^q}(K) \cap \Sigma_{L^p}(K)$, for $q \le r \le p$. That $\Sigma_{L^\infty}(K) = \Sigma_X(K)$ follows from Lemma 3.5. □

Our next result combines the above arguments with Theorem 2.1 and illustrates the application of that theorem. We use extensively the weak∗-convergence and topology on $L^\infty = (L^1)^*$ already used briefly in the proof of Lemma 3.5. For $(\psi_n) \subset L^\infty$, $\psi \in L^\infty$, we write $\psi_n \stackrel{w^*}{\to} \psi$ if (ψ_n) converges weak∗ to ψ, i.e., if

$$\int_{-\infty}^{+\infty} \psi_n(t)\phi(t)\,dt \longrightarrow \int_{-\infty}^{+\infty} \psi(t)\phi(t)\,dt, \quad \phi \in L^1.$$

A useful characterization of weak∗-convergence is that

$$\psi_n \xrightarrow{w*} \psi \Longleftrightarrow \sup_n \|\psi_n\|_\infty < \infty$$

and

$$\int_{-\infty}^{+\infty} \psi_n(t)\phi(t)\,dt \longrightarrow \int_{-\infty}^{+\infty} \psi(t)\phi(t)\,dt, \quad \phi \in C_0^\infty(\mathbf{R}).$$

We shall say that $V \subset L^\infty$ is weak∗-sequentially compact if every sequence in V has a subsequence converging weak∗ to an element of V.

Given $\kappa \in L^1$ and $V \subset L^\infty$, we will consider families of kernels $W = \{k_z : z \in V\}$ where $k_z(s,t) := \kappa(s-t)z(t)$, $s,t \in \mathbf{R}$. One significance of weak∗-convergence and weak∗ sequential compactness for our purposes is the relationship with the σ-convergence introduced in Section 2, expressed in the following lemmas.

Lemma 3.8 [**18**, Lemma 3.1]. *If $(z_n) \subset L^\infty$, $z \in L^\infty$, $z_n \stackrel{w*}{\to} z$, then $k_{z_n} \stackrel{\sigma}{\to} k_z$.*

Lemma 3.9 [**18**, Lemma 3.2]. *If $V \subset L^\infty$ is weak∗-sequentially compact, then $W = \{k_z : z \in V\}$ satisfies A_u and B_u and is σ-sequentially compact.*

Theorem 3.10. *Suppose $V \subset L^\infty$ is weak∗-sequentially compact, $R(V) = V$ and $T_r(V) = V$ for some $r \in \mathbf{R} \setminus \{0\}$. Suppose further that $\lambda \neq 0$, that $\lambda \notin \Sigma_X^p(\mathcal{K}M_z)$ for $z \in V$, and that for every $z \in V$ there exists $(z_n) \subset V$ such that $z_n \stackrel{w*}{\to} z$ and $(\lambda - \mathcal{K}M_{z_n})(X) = X$ for each n. Then, for $z \in V$, $(\lambda - \mathcal{K}M_z)^{-1} \in B(X)$ and $(\lambda - \mathcal{K}M_z)^{-1} \in B(L^p)$, $1 \le p \le \infty$. Further,*

$$\sup_{z \in V} \|(\lambda - \mathcal{K}M_z)^{-1}\|_X < \infty, \tag{3.10}$$

$$\sup_{\substack{z \in V \\ 1 \le p \le \infty}} \|(\lambda - \mathcal{K}M_z)^{-1}\|_{L^p} < \infty. \tag{3.11}$$

Proof. By Lemma 3.9, $W := \{k_z : z \in V\}$ satisfies conditions (i) and (ii) of Theorem 2.1, and W clearly also satisfies conditions (iii)–(v).

Thus, Theorem 3.1 applies to give that $(\lambda - \mathcal{K}M_z)^{-1} \in B(X)$, $z \in V$ and the bound (3.10). From Lemma 3.5, $(\lambda - \mathcal{K}M_z)^{-1} \in B(L^\infty)$, $z \in V$, and

$$\sup_{z\in V} \|(\lambda - \mathcal{K}M_z)^{-1}\|_{L^\infty} < \infty. \tag{3.12}$$

By Lemma 3.6 we obtain that $(\lambda - M_z\mathcal{K})^{-1} \in B(L^\infty)$, $z \in V$, and that these inverse operators are also uniformly bounded. Since

$$R(\lambda - M_z\mathcal{K})R = \lambda - M_{Rz}R\mathcal{K}R = \lambda - (\mathcal{K}M_{Rz})^T,$$

it follows that $(\lambda - (KM_{Rz})^T)^{-1} \in B(L^\infty)$ and, by (3.4), also $(\lambda - KM_{Rz})^{-1} \in B(L^1)$. Further, and since $R(V) = V$ and R is an isometrical isomorphism,

$$\begin{aligned} \sup_{z\in V} \|(\lambda - \mathcal{K}M_z)^{-1}\|_{L^1} &= \sup_{z\in V} \|(\lambda - \mathcal{K}M_{Rz})^{-1}\|_{L^1} \\ &= \sup_{z\in V} \|(\lambda - (\mathcal{K}M_{Rz})^T)^{-1}\|_{L^\infty} \\ &= \sup_{z\in V} \|(\lambda - M_z\mathcal{K})^{-1}\|_{L^\infty} < \infty. \end{aligned} \tag{3.13}$$

Thus $(\lambda - \mathcal{K}M_a)^{-1} \in B(L^\infty) \cap B(L^1)$ and, by Theorem 3.3, also $(\lambda - \mathcal{K}M_z)^{-1} \in B(L^p)$, $1 < p < \infty$. In view of (3.12), (3.13) and (3.3), the inequality (3.11) holds. □

We can obtain from the above result a statement about the spectra of the operators $\mathcal{K}M_z$.

Corollary 3.11. *Suppose $V \subset L^\infty$ is weak∗-sequentially compact, $R(V) = V$ and $T_r(V) = V$ for some $r \in \mathbf{R} \setminus \{0\}$. Suppose also that for every $\tilde{z} \in V$ there exists $(z_n) \subset V$ such that $z_n \stackrel{w*}{\to} \tilde{z}$ and*

$$\Sigma_X(\mathcal{K}M_{z_n}) \subset \{0\} \cup \bigcup_{z\in V} \Sigma^p_X(\mathcal{K}M_z).$$

Then, for $1 < r < \infty$, $p = 1, \infty$,

$$\bigcup_{z\in V} \Sigma_{L^r}(\mathcal{K}M_z) \subset \bigcup_{z\in V} \Sigma_{L^p}(\mathcal{K}M_z) = \bigcup_{z\in V} \Sigma_X(\mathcal{K}M_z) = \{0\} \cup \bigcup_{z\in V} \Sigma^p_X(\mathcal{K}M_z).$$

Proof. Examining the first part of the proof of Theorem 3.7, we see that the argument there applies to show that $\mathcal{K}(L^r) \neq L^r$, $1 \leq r \leq \infty$. Thus, for $z \in L^\infty$, $(\mathcal{K}M_z)(L^r) \neq L^r$, $1 \leq r \leq \infty$, so that $0 \in \Sigma_{L^r}(\mathcal{K}M_z)$, $1 \leq r \leq \infty$, $z \in V$. Hence, and as a corollary of the last theorem, we obtain that $\{0\} \subset \bigcup_{z\in V} \Sigma_{L^r}(\mathcal{K}M_z) \subset \{0\} \cup \bigcup_{z\in V} \Sigma_X^p(\mathcal{K}M_z)$, $1 \leq r \leq \infty$.

By Lemma 3.5, since k_z satisfies A and B, $\Sigma_{L^\infty}(\mathcal{K}M_z) = \Sigma_X(\mathcal{K}M_z)$. Examining the proof of Theorem 3.10, which uses Lemma 3.6 and (3.4), we see that the argument there is easily strengthened to obtain that, for $\lambda \neq 0$,

$$\bigcup_{z\in V} (\lambda - \mathcal{K}M_z)^{-1} \subset B(L^1) \Longleftrightarrow \bigcup_{z\in V} (\lambda - \mathcal{K}M_z)^{-1} \subset B(L^\infty).$$

Thus $\bigcup_{z\in V} \Sigma_{L^1}(\mathcal{K}M_z) = \bigcup_{z\in V} \Sigma_{L^\infty}(\mathcal{K}M_z)$. Putting these results together we obtain the stated corollary. □

We finish this section by illustrating the above theorems and those of Section 2 by using them to derive results for the following application, considered briefly in [**6**]. For $Q \subset \mathbf{C}$, let $L_Q := \{\phi \in L^\infty : \phi(s) \in Q$, for almost all $s \in \mathbf{R}\}$. It is shown in [**10**] that $V := L_Q$ is weak∗-sequentially compact if and only if Q is compact and convex. Whatever the choice of Q, clearly $R(V) = V$ and $T_r(V) = V$, $r \in \mathbf{R}$. Further, we can satisfy the remaining condition of Corollary 3.11 in a variety of ways, as discussed in [**6**]. For example, given $z \in V$, choose $(z_n) \subset V$ so that $z_n(s) = z(s)$, $|s| \leq n$ and so that $z_n(s) = q \in Q$ otherwise. Then, setting $\tilde{z}(s) = q$, $s \in \mathbf{R}$, $k_{z_n} - k_{\tilde{z}}$ satisfies A, B and C so that $\mathcal{K}M_{z_n} - \mathcal{K}M_{\tilde{z}}$ is compact on X. Hence, and since $\mathcal{K}M_{\tilde{z}} = q\mathcal{K}$ so that, see (1.4), $\Sigma_X(\mathcal{K}M_{\tilde{z}}) = \{0\} \cup \Sigma_X^p(\mathcal{K}M_{\tilde{z}})$, it follows that $\Sigma_X(\mathcal{K}M_{z_n}) \subset \Sigma_X^p(\mathcal{K}M_{z_n}) \cup \Sigma_X(\mathcal{K}M_{\tilde{z}}) \subset \{0\} \cup \Sigma_X^p(\mathcal{K}M_{z_n}) \cup \Sigma_X^p(\mathcal{K}M_{\tilde{z}}) \subset \{0\} \cup \bigcup_{z\in V} \Sigma_X^p(\mathcal{K}M_z)$.

We see that all the conditions of Theorem 3.10 and Corollary 3.11 are satisfied by the choice $V := L_Q$ if Q is compact and convex. In view of Lemma 3.9, it follows that, provided (1.6) holds for some $b > 1$, the conditions of Theorem 2.2 are also satisfied. Thus, applying Theorems 3.10 and 2.2, we have the following result, a significant extension of [**6**, Corollary 4.5], which only considers the cases $E = X$ and $E = X_a$.

Corollary 3.12. *Suppose $\kappa \in L^1$, $Q \subset \mathbf{C}$ is compact and convex, and $\lambda \neq 0$. Then* (i) *and* (ii) *are equivalent if E denotes one of X, L^1 or L^∞ and* (i) *implies* (ii) *if $E = L^p$ with $1 < p < \infty$.*

(i) *For every $z \in L_Q$, the equation*

$$\lambda\phi(s) = \int_{-\infty}^{+\infty} \kappa(s-t)z(t)\phi(t)\,dt, \quad s \in \mathbf{R}, \tag{3.14}$$

has only the trivial solution in X.

(ii) *For every $z \in L_Q$ the equation*

$$\lambda\phi(s) = \psi(s) + \int_{-\infty}^{+\infty} \kappa(s-t)z(t)\phi(t)\,dt, \quad s \in \mathbf{R}, \tag{3.15}$$

has exactly one solution $\phi \in E$ for every $\psi \in E$ and, for some constant $C > 0$ depending only on λ, κ and Q, $\|\phi\|_E \leq C\|\psi\|_E$.

If also $\kappa(s) = O(s^{-b})$ as $|s| \to \infty$ for some $b > 1$, then (i) *and* (ii) *are also equivalent for $E = X_a$, $0 < a \leq b$.*

As pointed out in [**6**], a boundary integral equation to which this result can be applied is obtained in [**11**] from a boundary value problem for the Helmholtz equation $\Delta u + k^2 u = 0$, $k > 0$, in the upper half-plane $U := \{(x_1, x_2) \in \mathbf{R}^2 : x_2 > 0\}$ with impedance boundary condition $\partial u/\partial x_2 + ik\beta u = f$ on $\partial U = \mathbf{R}$. (In the boundary condition, $\beta, f \in L^\infty$ are given boundary data.) The integral equation is of the form (3.15) with $\lambda = 1$, $z = i(1-\beta)$ and $\kappa \in L^1$ with $\kappa(s) = O(s^{-3/2})$ as $|s| \to \infty$. It is shown in [**11**] that the homogeneous equation (3.14) has only the trivial solution in X if $\beta \in L^\infty$ satisfies $\operatorname{ess.sup} \Re\beta > 0$. Thus if $Q = \{i(1-w) : w \in P\}$ with P a compact, convex subset of the right-hand complex plane, the conditions of Corollary 3.12 are satisfied and (i) in Corollary 3.12 holds. It follows that (ii) in Corollary 3.12 holds, with $E = L^p$, $1 \leq p \leq \infty$ and $E = X_a$, $0 \leq a \leq 3/2$.

We consider a related scattering problem in more detail in Section 4.

3.2 Results for general kernels. We turn now to a study of more general forms of kernel. To an extent our results on the solvability of (1.1) and its adjoint equation (3.1) in L^p, $1 \leq p \leq \infty$, are obtained using the same methods as in Section 3.1, in particular Theorems 3.1 and 3.2 relating the spectrum of K in L^p to that of K^T in L^q, and Lemma 3.5. However, we no longer have the possibility as in Section 3.1 of relating the spectra of K and K^T in the same space L^p via Lemma 3.6. With this key step in the argument missing we are obliged to make additional assumptions on k and k^T which, in fact, exclude the case $K = \mathcal{K}M_z$ unless $z \in X$. We also make an essential use of the result on solvability in weighted spaces obtained in Section 2.

Our main result is the following.

Theorem 3.13. *Suppose* $W \subset \mathcal{K}$, $W^T := \{k^T : k \in W\} \subset \mathcal{K}$, $\lambda \neq 0$, *and*

(i) W *satisfies* A'_u *and* (1.6) *holds for some* $b > 1$.

(ii) W *and* W^T *satisfy* B_u.

(iii) W *and* W^T *are* σ*-sequentially compact.*

(iv) $T_r^{(2)}(W) = W$ *for some* $r \in \mathbf{R} \setminus \{0\}$.

(v) *For every* $\tilde{k} \in W$ *there exists* $(k_n) \subset W$ *such that* $k_n \xrightarrow{\sigma} \tilde{k}$ *and, for each* n,

$$\lambda \notin \bigcup_{k \in W} \Sigma_X^p(K_k) \Longrightarrow \lambda \notin \Sigma_X(K_{k_n});$$

this statement also holds with W *replaced by* W^T.

(vi) $\lambda \notin \Sigma_X^p(K_k)$, $k \in W$.

Then $\lambda \notin \Sigma_X(K_k)$, $\lambda \notin \Sigma_{L^p}(K_k)$ *for* $k \in W \cup W^T$ *and* $1 \leq p \leq \infty$, *and*

$$\sup_{k \in W \cup W^T} \|(\lambda - K_k)^{-1}\|_X < \infty, \tag{3.16}$$

$$\sup_{\substack{1 \leq p \leq \infty \\ k \in W \cup W^T}} \|(\lambda - K_k)^{-1}\|_{L^p} < \infty. \tag{3.17}$$

Proof. Conditions (i)–(v) of Theorem 2.3 are satisfied by W. It follows from Theorem 2.3 that $\lambda \notin \Sigma_{X_a}(K_k)$, $k \in W$, $0 \leq a \leq b$, and then, from Lemma 3.5, that $\lambda \notin \Sigma_{L^\infty}(K_k)$, $k \in W$. Moreover, from Theorem 2.3 and Lemma 3.5 we have that

$$c := \sup_{k \in W} \|(\lambda - K_k)^{-1}\|_{L^\infty} = \sup_{k \in W} \|(\lambda - K_k)^{-1}\|_X < \infty.$$

Since, for $k \in W$, $\lambda \notin \Sigma_{X_b}(K_k)$ and $X_b \subset L^1$, it follows that $(\lambda - K_k)(L^1) \supset (\lambda - K_k)(X_b) = X_b$ which is dense in L^1. Since, by Theorem 3.2, $K_{k^T} \in B(L^\infty) = B((L^1)^*)$ is the adjoint of $K_k \in B(L^1)$ (and $K_k \in B(L^\infty)$ is the adjoint of $K_{k^T} \in B(L^1)$)) it follows from Theorem 3.1(ii) that $\lambda \notin \Sigma^p_{L^\infty}(K_k) = \Sigma^p_X(K_k)$, $k \in W^T$. We can now repeat the first part of the argument for we have established that conditions (i)–(v) of Theorem 2.3 are satisfied by W^T. In particular, it follows that $\lambda \notin \Sigma_X(K_k) = \Sigma_{L^\infty}(K_k)$, $k \in W^T$, and that

$$c^T := \sup_{k \in W^T} \|(\lambda - K_k)^{-1}\|_{L^\infty} = \sup_{k \in W^T} \|(\lambda - K_k)^{-1}\|_X < \infty.$$

The proof is completed by applying Theorem 3.1(iii) to deduce that $\lambda \notin \Sigma_{L^1}(K_k) \cup \Sigma_{L^1}(K_{k^T})$, $k \in W$, and that

$$\sup_{k \in W} \|(\lambda - K_k)^{-1}\|_{L^1} = c^T, \quad \sup_{k \in W^T} \|(\lambda - K_k)^{-1}\|_{L^1} = c.$$

It follows from Theorem 3.3 that also $\lambda \notin \Sigma_{L^p}(K_k)$, $k \in W \cup W^T$, $1 < p < \infty$, and that

$$\sup_{k \in W \cup W^T} \|(\lambda - K_k)^{-1}\|_{L^p} \leq \max(c, c^T),$$

for $1 \leq p \leq \infty$. $\square$

We can obtain, from the above result and some additional arguments, the following statement about the spectra of the operators K_k.

Corollary 3.14. *Suppose $W \subset \mathcal{K}$, $W^T := \{k^T : k \in W\} \subset \mathcal{K}$ and conditions* (i)–(iv) *of Theorem* 3.13 *are satisfied. Suppose also that, for every $\tilde{k} \in W$ and $k^* \in W^T$, there exist $(\tilde{k}_n) \subset W$ and $(k^*_n) \subset W^T$ such that $\tilde{k}_n \xrightarrow{\sigma} \tilde{k}$, $k^*_n \xrightarrow{\sigma} k^*$, and such that*

$$\Sigma_X(K_{\tilde{k}_n}) \subset \{0\} \cup \bigcup_{k \in W} \Sigma^p_X(K_k),$$
$$\Sigma_X(K_{k^*_n}) \subset \{0\} \cup \bigcup_{k \in W^T} \Sigma^p_X(K_k),$$

for each n. Then, for $1 < r < \infty$, $p = 1, \infty$, $V = W$, W^T,

$$\bigcup_{k \in V} \Sigma_{L^r}(K_k) \subset \bigcup_{k \in V} \Sigma_{L^p}(K_k) = \bigcup_{k \in V} \Sigma_X(K_k)$$

$$= \{0\} \cup \bigcup_{k \in W} \Sigma_X^p(K_k)$$

$$= \{0\} \cup \bigcup_{k \in W^T} \Sigma_X^p(K_k).$$

Proof. Since k and k^T satisfy A and B it follows from Lemma 3.5 that $0 \in \Sigma_X(K)$, $0 \in \Sigma_X(K^T)$. Hence, and from Theorem 3.13, it follows that, for $V = W$, W^T,

$$\lambda \notin \{0\} \cup \bigcup_{k \in V} \Sigma_X^p(K_k) \Longrightarrow \lambda \notin \bigcup_{k \in W \cup W^T} \Sigma_X(K_k)$$

$$\Longrightarrow \lambda \notin \{0\} \cup \bigcup_{k \in V} \Sigma_X^p(K_k^T).$$

Further, Theorem 3.13 implies that

$$\lambda \notin \{0\} \cup \bigcup_{k \in V} \Sigma_X^p(K_k) \Longrightarrow \lambda \notin \bigcup_{k \in V} \Sigma_{L^r}(K_k),$$

for $1 \le r \le \infty$. Moreover, by Lemma 3.5, $\Sigma_{L^\infty}(K) = \Sigma_X(K)$ and $\Sigma_{L^\infty}(K^T) = \Sigma_X(K^T)$, while by Theorem 3.1(iii) and Theorem 3.2, $\Sigma_{L^1}(K) = \Sigma_{L^\infty}(K^T)$ and $\Sigma_{L^1}(K^T) = \Sigma_{L^\infty}(K)$. Hence the corollary follows. □

We will illustrate the above general result by considering its application to a particular class of kernel which appears in the application in Section 4. Let $\tilde{\mathbf{R}}^2 := \{(s,t) \in \mathbf{R}^2 : s \neq t\}$, suppose that we are given $\kappa \in L^1$ and, for $z \in BC(\tilde{\mathbf{R}}^2)$, consider the kernel $\kappa_z \in \mathcal{K}$ defined by

$$\kappa_z(s,t) = \kappa(s-t) z(s,t), \quad s, t \in \mathbf{R}. \tag{3.18}$$

It is not difficult to see that kernels of this type satisfy conditions A' and B: see Lemma 3.16 below. Clearly,

$$\kappa_z^T(s,t) := \kappa_z(t,s) = \kappa(t-s) z(t,s)$$

is a kernel of the same type which thus also satisfies A' and B. For $(z_n) \subset BC(\tilde{\mathbf{R}}^2)$, $z \in BC(\tilde{\mathbf{R}}^2)$, we will write $z_n \stackrel{\tilde{s}}{\to} z$ if $\sup_n \|z_n\|_{BC(\tilde{\mathbf{R}}^2)} < \infty$ and $z_n(s,t) \to z(s,t)$ uniformly on compact subsets of $\tilde{\mathbf{R}}^2$. We will say that $V \subset BC(\tilde{\mathbf{R}}^2)$ is $\tilde{s}$-sequentially compact if every sequence $(z_n) \subset V$ has a subsequence (z_{n_m}) such that $z_{n_m} \stackrel{\tilde{s}}{\to} z \in V$. For $z \in BC(\tilde{\mathbf{R}}^2)$ define $z^T \in BC(\tilde{\mathbf{R}}^2)$ by

$$z^T(s,t) := z(t,s), \quad (s,t) \in \tilde{\mathbf{R}}^2.$$

Clearly,

$$z_n \stackrel{\tilde{s}}{\longrightarrow} z \Longrightarrow z_n^T \stackrel{\tilde{s}}{\longrightarrow} z^T. \tag{3.19}$$

The following lemmas enable the application of Theorem 3.13 to kernels of the type (3.18).

Lemma 3.15 [**18**, Lemma 3.4]. *If $z_n \stackrel{\tilde{s}}{\to} z$, then $\kappa_{z_n} \stackrel{\sigma}{\to} \kappa_z$.*

Lemma 3.16. *If $V \subset BC(\tilde{\mathbf{R}}^2)$ is $\tilde{s}$-sequentially compact, then $W := \{\kappa_z : z \in V\}$ satisfies A'_u and B_u and is σ-sequentially compact.*

Proof. Since V is $\tilde{s}$-sequentially compact, it is bounded and so A'_u holds. It is shown in [**18**, Lemma 3.5] that also B_u holds and W is σ-sequentially compact. □

Remark 3.17. If $V \subset BC(\tilde{\mathbf{R}}^2)$ is $\tilde{s}$-sequentially compact, then by (3.19) so is $V^T := \{z^T : z \in V\}$, and therefore, by Lemma 3.16, $W^T := \{\kappa_z^T : z \in V\}$ also satisfies A'_u and B_u and is σ-sequentially compact.

Combining Theorems 2.3 and 3.13 with Lemmas 3.15 and 3.16, and bearing in mind the above remark, we see that the following result holds.

Theorem 3.18. *Suppose $\kappa \in L^1$, $\lambda \neq 0$, $V \subset BC(\tilde{\mathbf{R}}^2)$, $W := \{\kappa_z : z \in V\}$ and*

(i) (1.6) *holds for some* $b > 1$.

(ii) V *is* $\tilde{s}$*-sequentially compact.*

(iii) $T_r^{(2)}(V) = V$ *for some* $r \in \mathbf{R} \setminus \{0\}$.

(iv) *For every* $\tilde{z} \in V$ *there exists* $(z_n) \subset V$ *such that* $z_n \xrightarrow{\tilde{s}} \tilde{z}$ *and, for each* n,

$$\lambda \notin \bigcup_{z \in V} \Sigma_X^p(K_{\kappa_z}) \Longrightarrow \lambda \notin \Sigma_X(K_{\kappa_{z_n}});$$

this statement also holds with κ_z, κ_{z_n} *replaced by* κ_z^T, $\kappa_{z_n}^T$.

(v) $\lambda \notin \Sigma_X^p(K_k)$, $k \in W$.

Then $\lambda \notin \Sigma_X(K_k)$, $\lambda \notin \Sigma_{L^p}(K_k)$ *and* $\lambda \notin \Sigma_{X_a}(K_k)$ *for* $k \in W \cup W^T$, $1 \le p \le \infty$, $0 \le a \le b$. *Further,* (3.16) *and* (3.17) *hold, and*

$$\sup_{k \in W \cup W^T} \|(\lambda - K_k)^{-1}\|_{X_a} < \infty,$$

for $0 \le a \le b$.

We will consider an important application of this theorem in the next section.

4. An application to rough surface scattering. We consider in this section the scattering of a time-harmonic wave field incident on an infinite boundary given as the graph of a bounded function f. More precisely, denote by $C^{1,1}(\mathbf{R})$ the Hölder space

$$C^{1,1}(\mathbf{R}) := \Big\{\phi \in C^1(\mathbf{R}) : \|\phi\|_{C^{1,1}(\mathbf{R})} := \|\phi\|_\infty + \|\phi'\|_\infty + \sup_{s,t \in \mathbf{R}, s \ne t} \frac{|\phi'(s) - \phi'(t)|}{|s-t|} < \infty\Big\}.$$

Given $f \in C^{1,1}(\mathbf{R})$, define the domain $D := \{x = (x_1, x_2) \in \mathbf{R}^2 : x_2 > f(x_1)\}$, and let $\Gamma := \partial D$ denote the boundary of D. We assume that the time-harmonic incident field u^i is a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$

with wavenumber $k > 0$ and that u^i is continuous up to the boundary Γ with $u^i|_\Gamma$ bounded and restrict our attention to the case where the total field vanishes on the boundary. Then the scattered field u has to satisfy the Helmholtz equation in D and the boundary condition $u = -u^i$ on Γ. Set, for $h \in \mathbf{R}$, $\Gamma_h := \{x \in \mathbf{R}^2 : x_2 = h\}$ and $U_h := \{x \in \mathbf{R}^2 : x_2 > h\}$. In order to ensure uniqueness of solution to this problem, we further require that the scattered field be bounded in the horizontal strip $D \setminus U_h$ for every $h > 0$ and that it satisfy the *upward propagating radiation condition* proposed in [**11, 17**]: that, for some $h > \sup_{x_1 \in \mathbf{R}} f(x_1)$ and $\phi \in L^\infty(\Gamma_h)$,

$$u(x) = 2\int_{\Gamma_h} \frac{\partial \Phi(x,y)}{\partial y_2} \phi(y)\, ds(y), \quad x \in U_h, \tag{4.1}$$

where $\Phi(x,y) := (i/4)H_0^{(1)}(k|x-y|)$, $x, y \in \mathbf{R}^2$, $x \neq y$, is the free field Green's function for the Helmholtz equation and $H_0^{(1)}$ denotes the Hankel function of the first kind and of order 0.

Thus the rough surface scattering problem we are considering is a special case of the following Dirichlet boundary value problem

Problem 4.1. *Given $g \in BC(\Gamma)$, determine $u \in C^2(D) \cap C(\overline{D})$ such that*

1. $\Delta u + k^2 u = 0$ in D,
2. $u = g$ on Γ,
3. *u is bounded in $D \setminus U_h$ for all $h > 0$,*
4. *u satisfies the* UPRC (4.1).

It has been shown in [**17, 14**] that Problem 4.1 admits a unique solution for any boundary function $g \in BC(\Gamma)$.

We will now study an equivalent boundary integral formulation of Problem 4.1 derived in [**14**], applying the results of the preceding sections. Let $G(x,y) := \Phi(x,y) + \Phi(x,y') + P(x-y')$ for $x, y \in \overline{U}_0$, $x \neq y$, where $y = (y_1, y_2)$, $y' = (y_1, -y_2)$ and

$$P(x) := \frac{e^{ik|x|}}{\pi} \int_0^\infty \frac{t^{-1/2} e^{-k|x|t}(1 + \gamma(1+it))}{\sqrt{t-2i}\,(t - i(1+\gamma))^2}\, dt, \quad x \in \overline{U}_0,$$

with $\gamma = x_2/|x|$. It follows from this definition, see [**11, 12**], that $G(x,y)$ is the Green's function for the operator $\Delta + k^2$ in the upper half-plane U_0 which satisfies the impedance boundary condition

$$\frac{\partial G(x,y)}{\partial x_2} + ikG(x,y) = 0, \quad x \in \Gamma_0,\ y \in \overline{U}_0,\ x \neq y.$$

Moreover, it was shown in [**13**] that $G(x,y)$ exhibits a more rapid decay than $\Phi(x,y)$ as $|x_1 - y_1| \to \infty$ with x_2, y_2 bounded, as expressed in the bounds

(4.2)

$$|G(x,y)|, \quad |\nabla_y G(x,y)| \leq C\frac{(1+x_2)(1+y_2)}{|x-y|^{3/2}}, \quad x, y \in \overline{U}_0,\ x \neq y,$$

where the constant $C > 0$ only depends on k.

It was proposed in [**13**] to seek a solution to Problem 4.1 in the form of a double layer potential

$$u(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(y)} \psi(y)\, ds(y), \quad x \in D, \tag{4.3}$$

for some $\psi \in BC(\Gamma)$ where $\nu(y)$ denotes the unit normal vector at $y \in \Gamma$ pointing out of D. Note that, by the bound (4.2), the double layer potential exists as an improper integral for all $x \in D$. Moreover, it was shown [**17**, Theorem 4.2] that u defined by (4.3) is a solution to Problem 4.1 provided $\psi \in BC(\Gamma)$ satisfies the boundary integral equation

$$\psi(x) - 2\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(y)} \psi(y)\, ds(y) = -2g(x), \quad x \in \Gamma. \tag{4.4}$$

Defining $\phi, \gamma \in BC(\mathbf{R})$ by

$$\phi(s) := \psi((s, f(s))),\ \gamma(s) := -2g((s, f(s))), \quad s \in \mathbf{R},$$

we find (4.4) to be equivalent to the integral equation on the real line,

$$\phi(s) - 2\int_{-\infty}^{\infty} \frac{\partial G(x,y)}{\partial \nu(y)} \phi(t)\sqrt{1 + f'(t)^2}\, dt = \gamma(s), \quad s \in \mathbf{R}, \tag{4.5}$$

where $x = (s, f(s))$, $y = (t, f(t))$.

Many of the results of the earlier sections of this paper apply not to a single integral operator but to families of such operators whose kernels satisfy conditions of translation invariance, Assumptions A_u and B_u, and certain other conditions. To make use of these results, for $c, M > 0$, we introduce the set of surface functions $B_{c,M}$, defined by

$$B_{c,M} := \{f \in C^{1,1}(\mathbf{R}) : \inf f \geq c, \|f\|_{C^{1,1}(\mathbf{R})} \leq M\}.$$

For $f \in B_{c,M}$ we define the kernel k_f by

$$k_f(s,t) := 2\frac{\partial G(x,y)}{\partial \nu(y)}\sqrt{1+f'(t)^2}, \quad s,t \in \mathbf{R}, s \neq t,$$

where $x = (s, f(s)), y = (t, f(t))$ and define the integral operator K_f by

$$K_f\phi(s) := \int_{-\infty}^{\infty} k_f(s,t)\phi(t)\,dt, \quad s \in \mathbf{R},\ \phi \in X.$$

The integral equation (4.5) can then be rewritten in operator notation as

$$(1 - K_f)\phi = \gamma. \tag{4.6}$$

From previous studies, we have the following results.

Lemma 4.2 [**14**, Theorem 5.1]. *The integral equation* (4.6) *has at most one solution in* X*.*

Lemma 4.3 [**13**, Lemma 5.1]. *There exists* $C > 0$*, depending only on* c, M *and the wavenumber* k *such that, for all* $f \in B_{c,M}$*,*

$$|k_f(s,t)| \leq C\kappa(s-t), \quad s,t \in \mathbf{R},\ s \neq t, \tag{4.7}$$

where $\kappa(s) := (1+|s|)^{-3/2}$, $s \in \mathbf{R}$.

Defining

$$z_f(s,t) := \frac{k_f(s,t)}{\kappa(s-t)}, \quad f \in B_{c,M},\ s,t \in \mathbf{R},\ s \neq t,$$

we have that $k_f(s,t) = \kappa(s-t)z_f(s,t)$. By Lemma 4.3 and since $k_f(s,t)$ is clearly continuous for $s \neq t$, we have that $z_f \in BC(\tilde{\mathbf{R}}^2)$. We set

$$V := \{z_f : f \in B_{c,M}\}, \quad W := \{k_f : f \in B_{c,M}\}.$$

Lemma 4.4 [**14**, Lemma 4.6] and [**18**, Lemma 4.6].

(a) *Every sequence* $(f_n) \subset B_{c,M}$ *has a subsequence* (f_{n_m}) *such that* $f_{n_m} \xrightarrow{s} f$, $f'_{n_m} \xrightarrow{s} f'$, *with* $f \in B_{c,M}$.

(b) *Suppose that* $(f_n) \subset B_{c,M}$ *and that* $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$ *with* $f \in B_{c,M}$. *Then* $z_{f_n} \xrightarrow{\tilde{s}} z_f$.

From Lemma 4.4, it follows that the set V is $\tilde{s}$-sequentially compact. Moreover, for $r \in \mathbf{R}$ it holds that $T_r^{(2)}(V) = V$ since $T_r(B_{c,M}) = B_{c,M}$. By Lemma 4.3, (1.6) holds with $b = 3/2$. Finally $1 \notin \Sigma_X^p(K_f)$, for $f \in B_{c,M}$ by Lemma 4.2. Thus, to apply Theorem 3.18, with $\lambda = 1$, only condition (iv) remains to be shown.

For $f \in B_{c,m}$, define

$$\bar{f}(s) := \frac{\sup f + \inf f}{2}, \quad s \in \mathbf{R}.$$

Note that $\bar{f} \in B_{c,M}$. For all $A > 0$ sufficiently large, there exists a sequence $(f_n) \subset B_{c,M}$ such that

$$f_n(s) = \begin{cases} f(s) & |s| \leq n, \\ \bar{f}(s) & |s| \geq n + A. \end{cases}$$

Clearly $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$, so that $z_{f_n} \xrightarrow{\tilde{s}} z_f$, by Lemma 4.4(b).

Moreover, $z_{\bar{f}}(x,t) = \tilde{z}(s-t)$ for some $\tilde{z} \in BC(\mathbf{R} \setminus \{0\})$ so that the operator $K_{\bar{f}}$ is a convolution operator and, by (1.4), $\{0\} \cup \Sigma_X^p(K_{\bar{f}}) = \Sigma_X(K_{\bar{f}})$. Hence, $1 \notin \Sigma_X(K_{\bar{f}})$ so that $1 - K_{\bar{f}}$ is Fredholm of index zero. Furthermore, since V is $\tilde{s}$-sequentially compact, it follows from Lemma 3.16 that $l_n := k_{f_n} - k_{\bar{f}}$ satisfies A and B. In view of the bound (4.2) and since $l_n(s,t) = 0$, if $|s| \geq n + A$ and $|t| \geq n + A$, it is easy to see that l_n also satisfies C. Thus (see Section 2), $K_{f_n} - K_{\bar{f}}$ is compact so that $1 - K_{f_n} = 1 - K_{\bar{f}} - (K_{f_n} - K_{\bar{f}})$ is also Fredholm of index zero. Hence, $1 \notin \Sigma_X(K_{f_n})$ for each n. Furthermore, if $1 \notin \bigcup_{f\in B_{c,M}} \Sigma_X^p(K_f^T)$, then, repeating the above argument, we can show that $1 \notin \Sigma_X(K_{f_n}^T)$ for each n. Thus we have shown that assumption (iv) of Theorem 3.18 holds. An application of this theorem now yields the following corollary.

Corollary 4.5. *For $1 \leq p \leq \infty$, the integral equation* (4.6) *has exactly one solution $\phi \in L^p$ for every $\gamma \in L^p$ and $f \in B_{c,M}$. There exists a constant $c^* > 0$ depending only on c, M and the wavenumber k such that $\|\phi\|_p \leq c^*\|\gamma\|_p$ for $1 \leq p \leq \infty$, $\gamma \in L^p$, $f \in B_{c,M}$. If $\gamma \in X_a$, for some $a \in [0, 3/2]$, it holds that $\phi \in X_a$. Moreover for $0 \leq a \leq 3/2$, there exists a constant $C_a > 0$, such that for all $f \in B_{c,M}$, $\gamma \in X_a$, the solution ϕ of* (4.6) *satisfies*

$$|\phi(s)| \leq C_a(1 + |s|)^{-a} \sup_{t\in\mathbf{R}} |(1 + |t|)^a\gamma(t)|, \quad s \in \mathbf{R}.$$

The assertions of this corollary also hold with K_f replaced by K_f^T in equation (4.6)*.*

For the original boundary value problem, Problem 4.1, we have as a corollary the following existence result [**14**, Theorem 5.3].

Corollary 4.6. *There exists a unique solution to Problem* 4.1*.*

However, the ansatz (4.3) is not the only possibility to seek the solution to the scattering problem. We can also derive an integral equation from Green's theorem. It is convenient for this purpose to make the additional assumption that the incident field satisfies the Helmholtz equation in the half-plane $U_{-\varepsilon}$ for some $\varepsilon > 0$ and further to suppose that u^i is bounded in the strip $U_{-\varepsilon} \setminus U_H$ for some $H > \sup f$. Introducing the total field $u^t = u^i + u$, since $u^t = 0$ on Γ and Γ is Lyapunov, it follows from regularity estimates up to the boundary for solutions to elliptic equations [**20**] that $u^t \in C^2(D) \cap C^1(\overline{D})$. Further, since u^t is bounded in $D \setminus U_h$, arguing exactly as in the proof of [**17**, Theorem 3.1], we can show that, for $0 < \alpha < 1$, $h' < H$, there exists a constant $C > 0$ such that

$$|\nabla u^t(x)| \leq C(x_2 - f(x_1))^{\alpha-1}, \quad x \in D \setminus U_{h'}. \tag{4.8}$$

Also, arguing as in the proof of [**17**, Theorem 3.2], we have that

$$\sup_{n\in\mathbf{Z}} \int_{\Gamma(n,n+1)} \left|\frac{\partial u^t}{\partial \nu}\right|^2 ds < \infty, \tag{4.9}$$

where $\Gamma(n, n+1) := (y = (y_1, y_2) \in \Gamma : n \leq y_1 < n+1\}$, $n \in \mathbf{Z}$. These estimates are not the strongest possible bounds, as we shall see shortly, but are sufficient for the following derivation of the integral equation formulation.

For $A > 0$ and $H > h > \sup f$, we can apply Green's theorem in the domain $D_{A,h} := \{y = (y_1, y_2) \in D : |y_1| < A, y_2 < h\}$, to obtain

$$u^t(x) = \int_{\partial D_{A,h}} \left\{G(x,y)\frac{\partial u^t}{\partial \nu}(y) - \frac{\partial G(x,y)}{\partial \nu(y)} u^t(y)\right\} ds(y), \quad x \in D_{A,h}.$$

In view of the bounds (4.2) and (4.8) and the boundary condition $u^t = 0$ on Γ, it then follows, taking the limit $A \to \infty$, that

$$\begin{aligned} u^t(x) = \int_{\Gamma_h} &\left\{G(x,y)\frac{\partial u^t}{\partial \nu}(y) - \frac{\partial G(x,y)}{\partial \nu(y)} u^t(y)\right\} ds(y) \\ &+ \int_{\Gamma} G(x,y)\frac{\partial u^t}{\partial \nu}(y)\, ds(y), \quad x \in D_{\infty,h}, \end{aligned} \tag{4.10}$$

where the normal ν to Γ_h is pointing upwards, and the integral over Γ is well defined in view of the bounds (4.2) and (4.9). Using the fact that the scattered field u satisfies the UPRC, the terms in the first integral in this equation containing u are seen to vanish by the equivalence of (i) and (v) in [**16**, Theorem 2.9]. Thus we are left with the expression

$$u^t(x) = \tilde{u}(x) + \int_{\Gamma} G(x,y)\frac{\partial u^t}{\partial \nu}(y)\, ds(y), \quad x \in D \setminus \overline{U}_h, \tag{4.11}$$

where

$$\tilde{u}(x) := \int_{\Gamma_h} \left\{G(x,y)\frac{\partial u^i}{\partial \nu}(y) - \frac{\partial G(x,y)}{\partial \nu(y)} u^i(y)\right\} ds(y), \quad x \in U_0 \setminus \overline{U}_h.$$

Let $u^r \in C^2(U) \cap C(\overline{U})$ denote the solution of the Helmholtz equation in U which is bounded in the strip $U \setminus U_h$ for every $h > 0$ and satisfies the radiation condition (4.1) and the impedance boundary condition

$$\frac{\partial u^r}{\partial x_2} + iku^r = -\left(\frac{\partial u^i}{\partial x_2} + iku^i\right)$$

on Γ_0, in the weak sense of [**11**]. By [**11**] the unique solution to this boundary value problem is

$$u^r(x) = \int_{\Gamma_0} G(x,y)\left(\frac{\partial u^i}{\partial y_2} + iku^i(y)\right) ds(y), \quad x \in \overline{U}. \tag{4.12}$$

By a similar argument to that used to derive (4.10), but applying Green's theorem in the region between Γ_h and Γ_0, it follows that $\tilde{u}(x) = u^i(x) + u^r(x)$, $x \in U_0 \setminus U_h$. Substituting in (4.11), we obtain that

$$u^t(x) = u^i(x) + u^r(x) + \int_\Gamma G(x,y)\frac{\partial u^t}{\partial \nu}(y)\, ds(y), \tag{4.13}$$

for $x \in D \setminus U_h$ and, by analytic continuation, this equation holds throughout D. Note that, for the incident fields commonly of interest, a more explicit expression than (4.12) can be given for u^r. In particular, if u^i is the incident plane wave $u^i(x) = \exp(ikx \cdot d)$ for some unit vector $d = (d_1, d_2)$ with $d_2 < 0$, then $u^r(x) = R\exp(ikx \cdot d')$ where $d' = (d_1, -d_2)$ and R is the reflection coefficient $R = (-d_2 - 1)/(-d_2 + 1)$.

Using standard properties of the acoustic double layer potential [**19**] together with the bound (4.2) on the fundamental solution, it is not difficult to see from (4.13) that the normal derivative $\partial u^t/\partial\nu$ satisfies the following integral equation on Γ:

(4.14)

$$\frac{\partial u^t}{\partial \nu}(x) - 2\int_\Gamma \frac{\partial G(x,y)}{\partial \nu(x)}\frac{\partial u^t}{\partial \nu}(y)\, ds(y) = 2\left(\frac{\partial u^i}{\partial \nu}(x) + \frac{\partial u^r}{\partial \nu}(x)\right), \quad x \in \Gamma.$$

This equation is equivalent to the integral equation

$$\phi(s) - \int_{-\infty}^{\infty} l_f(s,t)\phi(t)\, dt = \rho(s), \quad s \in \mathbf{R}, \tag{4.15}$$

on the real line, where $\phi(s) := (\partial u^t/\partial \nu)((s, f(s))), \rho(s) := 2(\partial u^i/\partial\nu + \partial u^r/\partial\nu)((s, f(s)))$, $s \in \mathbf{R}$, and we have introduced the kernel

$$l_f(s,t) := 2\frac{\partial G(x,y)}{\partial \nu(x)} w_f(t),$$

with $x = (s, f(s))$, $y = (t, f(t))$ and $w_f(t) := \sqrt{1 + f'(t)^2}$. From (4.9) it follows that $\sup_{n\in\mathbf{Z}} \int_n^{n+1} |\phi(t)|^2\, dt < \infty$. Thus, and since $l_f(s,t) = k_f(t,s)w_f(t)/w_f(s)$ so that, by Lemma 4.3, $|l_f(s,t)| \le C(1 + |s - t|)^{-3/2}$, it follows from (4.15) and the Cauchy-Schwarz inequality that $\phi \in L^\infty(\mathbf{R})$. Defining the integral operator $\tilde{K}_f$ by

$$\tilde{K}_f \phi(s) := 2 \int_{-\infty}^{\infty} \frac{\partial G(x,y)}{\partial \nu(y)} \phi(t)\, dt, \quad s \in \mathbf{R},$$

with $x = (s, f(s)), y = (t, f(t))$, we easily verify that $K_f = \tilde{K}_f M_{w_f}$ where M_{w_f} is the operation of multiplication by w_f and that equation (4.15) can be written in operator notation as

$$(1 - \tilde{K}_f^T M_{w_f})\phi = \rho. \tag{4.16}$$

However, by Lemma 3.6, it follows that (4.16) is solvable if the operator $1 - M_{w_f}\tilde{K}_f^T = 1 - K_f^T$ is invertible. This result has already been shown in Corollary 4.5. Thus, combining Lemma 3.6 with Corollary 4.5, we have the following final result, a statement of well-posedness for our second boundary integral equation formulation of the scattering problem.

Corollary 4.7. *For $1 \le p \le \infty$, the integral equation* (4.15) *has exactly one solution $\phi \in L^p$ for every $\rho \in L^p$ and $f \in B_{c,M}$. There exists a constant $c^* > 0$, depending only on c, M and the wavenumber k such that $\|\phi\|_p \le c^* \|\rho\|_p$, for $1 \le p \le \infty$, $\rho \in L^p$, $f \in B_{c,M}$. If $\rho \in X_a$, for some $a \in [0, 3/2]$, it holds that $\phi \in X_a$. Moreover, for $0 \le a \le 3/2$, there exists a constant $C_a > 0$ such that, for all $f \in B_{c,M}$, $\rho \in X_a$, the solution ϕ of* (4.15) *satisfies*

$$|\phi(s)| \le C_a (1 + |s|)^{-a} \sup_{t\in\mathbf{R}} |(1 + |t|)^a \rho(t)|, \quad s \in \mathbf{R}.$$

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