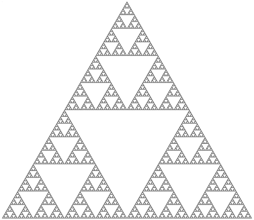
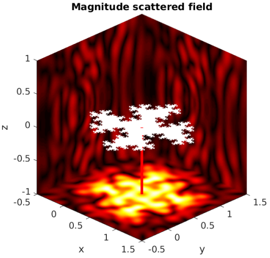
A Hausdorff-measure boundary element method for scattering by fractal screens
I: Numerical Analysis



Simon Chandler-Wilde

Department of Mathematics and Statistics
University of Reading, UK



With: António Caetano (Aveiro), Andrew Gibbs, Dave Hewett (UCL), & Andrea Moiola (Pavia)

Waves 2022, July 2022

Time-harmonic acoustic scattering by a fractal screen

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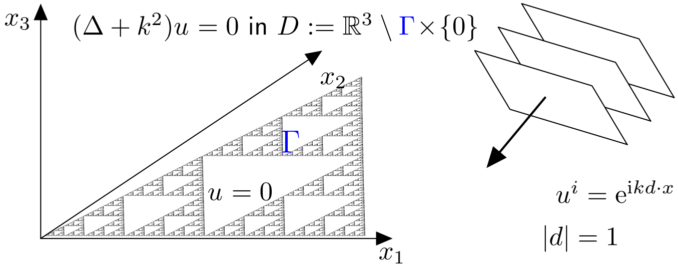

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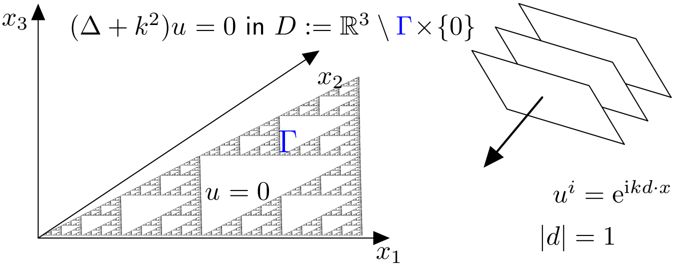

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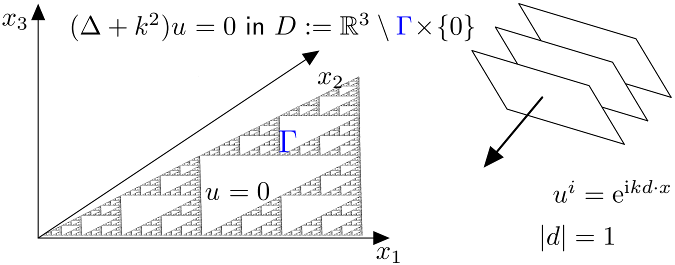

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$$[\partial u/\partial n] \in H_\Gamma^{-1/2} = \{\phi \in H^{-1/2}(\mathbb{R}^2) : \operatorname{supp}\phi \subset \Gamma\}.$$

This satisfies the boundary integral equation (BIE)

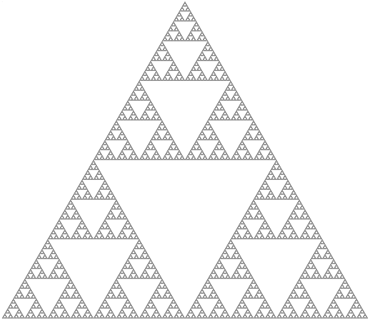
$$S[\partial u/\partial n] = u^i|_\Gamma,$$

where S is the single-layer boundary integral operator.

This talk

We introduce a new "conforming" discretization where

- the BEM basis functions are supported in Γ
- integration is carried out with respect to Hausdorff (not Lebesgue) measure – see Andrew Gibbs' talk (next!) for details of numerical quadrature



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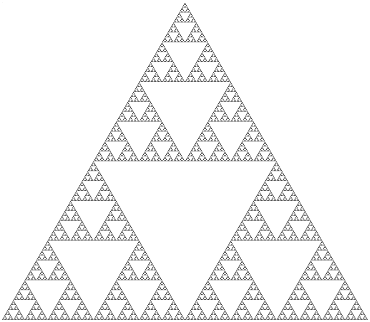

Convergence analysis can follow the standard Lax-Milgram Lemma/Céa's Lemma approach

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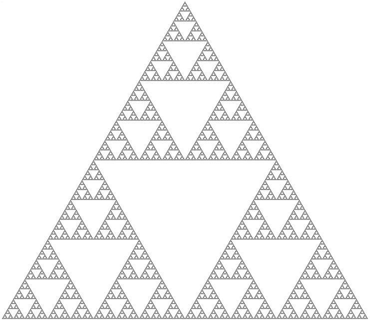
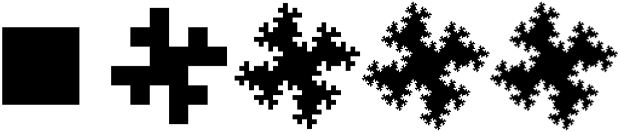
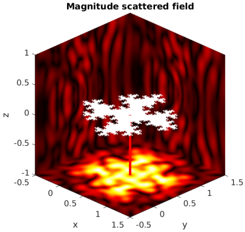


Convergence analysis can follow the standard Lax-Milgram Lemma/Céa's Lemma approach

Convergence rates can be obtained under natural solution regularity assumptions

Why fractals?





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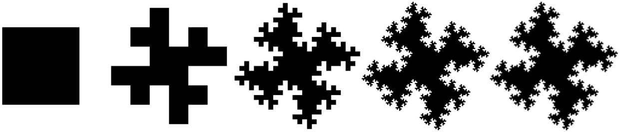

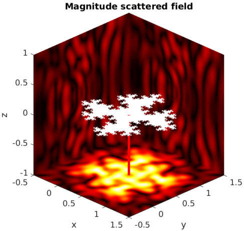

Fractals are an obvious mathematical model for the **multiscale roughness** possessed by many naturally-occuring and man-made scatterers.

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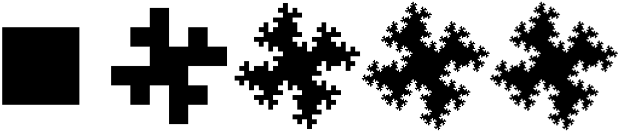
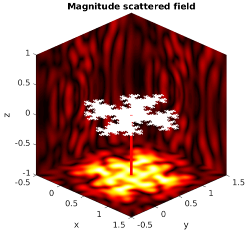




Fractals are an obvious mathematical model for the **multiscale roughness** possessed by many naturally-occuring and man-made scatterers.

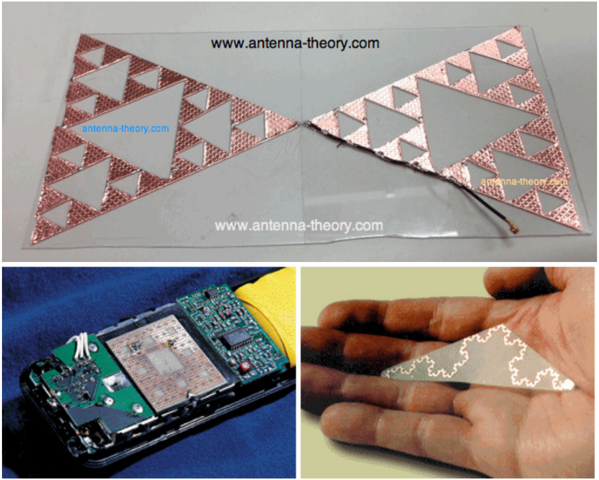
They are also a **rich source of mathematical challenges** that are stimulating exciting new research in modelling, function spaces and numerical analysis.

M. V. Berry, "Diffractals", J. Phys. A., 1979 - "*a new regime in wave physics*"

U. Mosco, 2013 - "*introducing fractal constructions into the classic theory of PDEs opens a vast new field of study, both theoretically and numerically*", "*this new field has been only scratched*"

Applications

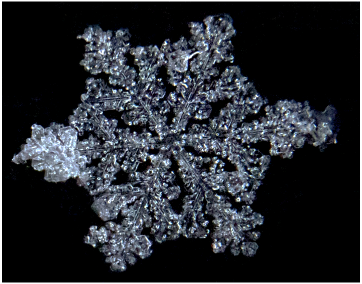
Fractal antennas - wideband/multiband performance from a compact design



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

Applications

Scattering by ice crystals in atmospheric physics
e.g. Chris Westbrook (Reading Meteorology)



Fractal apertures in laser optics
e.g. James Christian (Salford Physics)

Preliminaries: Hausdorff measure and dimension

For $E \subset \mathbb{R}^n$ and $d \geq 0$

$$\mathcal{H}^d(E) \in [0, \infty) \cup \{\infty\},$$

is the usual d-**dimensional Hausdorff (outer) measure of** E.

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NOTE: $\dim_{\mathrm{H}}(\text{point}) = 0$, $\dim_{\mathrm{H}}(\text{line}) = 1$, $\dim_{\mathrm{H}}(\text{area}) = 2$, etc., but non-integer dimensions are also possible!

Preliminaries: Iterated function systems (IFS)

We'll assume $\Gamma = \cup_{m=1}^{M} s_m(\Gamma)$ is the attractor of an **IFS of contracting similarities** $\{s_1, s_2, \ldots, s_M\}$ (with $M \geq 2$), so each $s_m : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$|s_m(x) - s_m(y)| = \rho_m |x - y|, \quad x, y \in \mathbb{R}^2,$$

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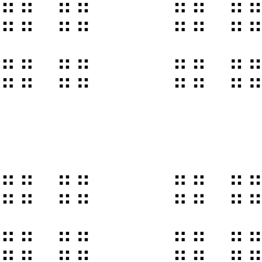
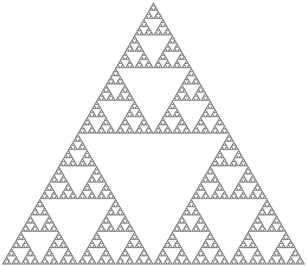


Middle third Cantor dust
$M = 4$, $\rho_m = 1/3$,
$\dim_{\mathrm{H}} \Gamma = \log 4 / \log 3$



Sierpinski triangle
$M = 3$, $\rho_m = 1/2$,
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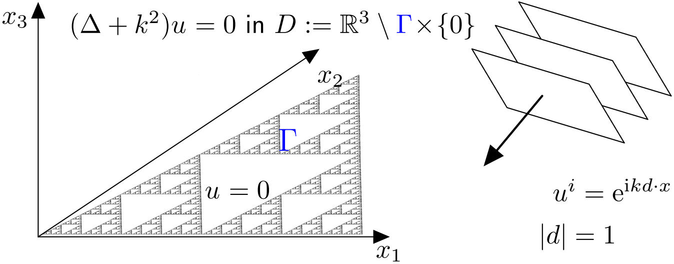

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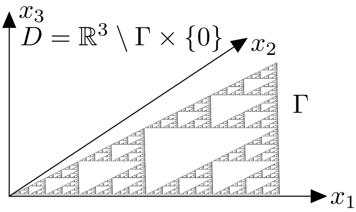
BIE formulation



Theorem (C-W/Hewett 2018)

The sound-soft Helmholtz BVP has unique solution

$$u(x) = u^i(x) - \mathcal{S}\phi(x), \qquad x \in D,$$

where $\phi = \partial_{\mathrm{n}}^+ u - \partial_{\mathrm{n}}^- u \in H_\Gamma^{-1/2}$ is the unique solution of the BIE

$$S\phi = u^i|_\Gamma.$$

Here $\mathcal{S} : H_\Gamma^{-1/2} \to C^2(D) \cap W^{1,\mathrm{loc}}(\mathbb{R}^{n+1})$ is the single-layer potential operator, and $S : H_\Gamma^{-1/2} \to (H_\Gamma^{-1/2})^*$ is the single layer BIO

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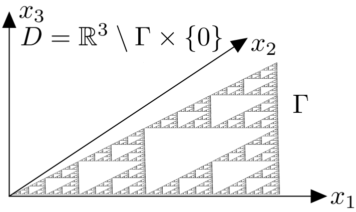

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Variational formulation and Galerkin BEM

The operator $S : H_\Gamma^{-1/2} \to (H_\Gamma^{-1/2})^*$ is continuous and coercive (strongly elliptic) - see C-W/Hewett 2015.

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$$\text{find } \phi \in V_N \text{ s.t. } \langle S\phi, \psi \rangle_{H^{1/2} \times H^{-1/2}} = \langle u^i, \psi \rangle_{H^{1/2} \times H^{-1/2}}, \quad \forall \psi \in V_N,$$

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$$V_N = \{ f \mathcal{H}^d|_\Gamma \} \subset H_\Gamma^{-1/2}$$

where $d = \dim_{\mathrm{H}} \Gamma$ and f is a piecewise constant function on a "mesh" of Γ.

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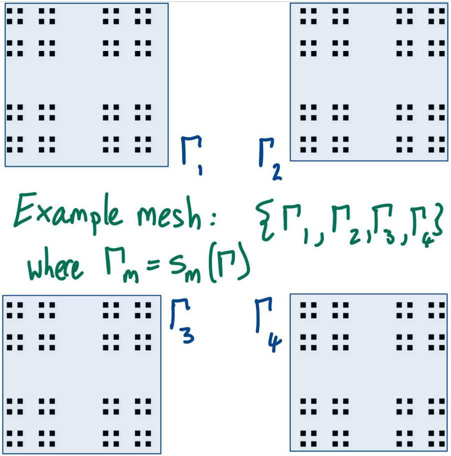
A "**mesh**" is a set $\{T_j\}_{j=1}^N$ of $\mathcal{H}^d$-measurable subsets of Γ (the "**elements**") with $\mathcal{H}^d(T_j) > 0$ for each j, $\mathcal{H}^d(T_j \cap T_{j'}) = 0$ for $j \neq j'$, and $\Gamma = \bigcup_{j=1}^N T_j$.

Iterated function systems

When Γ is the fractal attractor of an IFS we will choose each "**mesh element**" T_j to be one of the "**components**" of Γ, defined by

$$\Gamma_{m_1,m_2,\dots,m_p} := s_{m_1} \circ \cdots \circ s_{m_p}(\Gamma), \quad \text{with } m_j \in \{1,\dots,M\}, \quad j = 1,\dots,p,$$

each of which is a scaled version of Γ



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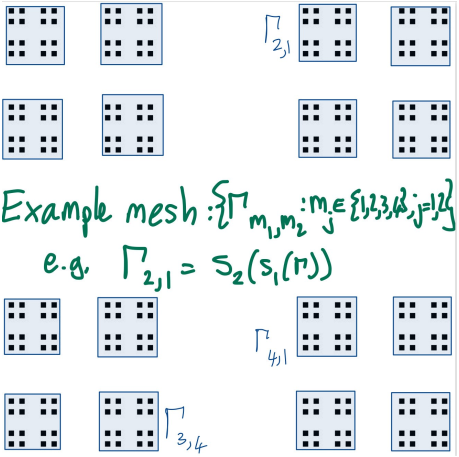
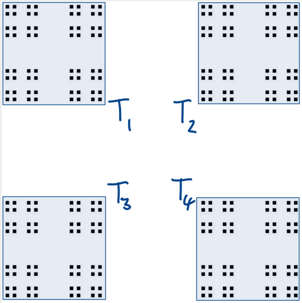


Galerkin matrix entries



Using the canonical basis $f^1, \ldots, f^N$ for V_N, with each $f^j|_{T_j} = 1$ and $f^j|_{\Gamma \setminus T_j} = 0$, the Galerkin BEM matrix has entries

$$A_{ij} = \int_{T_i} \int_{T_j} \Phi(x, y) \,\mathrm{d}\mathcal{H}^d(y) \mathrm{d}\mathcal{H}^d(x),$$

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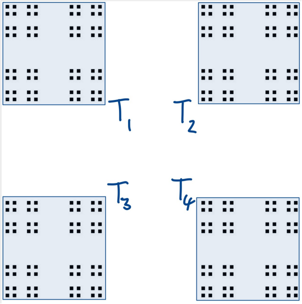

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Practical implementation requires singular quadrature with respect to Hausdorff measure - see Andrew Gibbs' talk!

Main convergence result

Specify $h > 0$ and choose as the elements of our mesh all those components with diameter $\leq h$ whose "parent" has diameter $> h$.

Theorem

Suppose $\Gamma_1, \ldots, \Gamma_M$ disjoint, $1 < d = \dim_{\mathrm{H}}\Gamma < 2$. Suppose exact BIE solution $\phi \in H^s_\Gamma$ with $-1/2 < s < \frac{d}{2} - 1$. Then Hausdorff BEM solution ϕ_N satisfies

$$\|\phi - \phi_N\|_{H_\Gamma^{-1/2}} \lesssim h^{s+1/2} \|\phi\|_{H^s_\Gamma}, \qquad h > 0.$$

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If $d \leq 1$ then $\phi = 0$, incident wave doesn't "see" Γ – see C-W/Hewett 2018

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$\frac{d}{2} - 1$ is a **hard upper limit**; $H^s_\Gamma = \{0\}$ for $s \geq \frac{d}{2} - 1$ – see Hewett/Moiola 2017

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Proof uses Céa's lemma plus

- Trace theorems from $H^s(\mathbb{R}^2)$ to $L^2(\Gamma; \mathcal{H}^d)$, and relationship between H^s_Γ spaces and "trace spaces" on Γ – Caetano/Hewett/Moiola 2021
- Haar-type-wavelet characterisations of trace spaces on fractals – Jonsson 1998

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$\frac{d}{2} - 1$ is a **hard upper limit**; $H^s_\Gamma = \{0\}$ for $s \geq \frac{d}{2} - 1$ – see Hewett/Moiola 2017

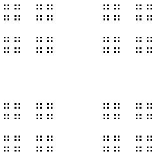
Proof uses Céa's lemma plus

- Trace theorems from $H^s(\mathbb{R}^2)$ to $L^2(\Gamma; \mathcal{H}^d)$, and relationship between H^s_Γ spaces and "trace spaces" on Γ – Caetano/Hewett/Moiola 2021
- Haar-type-wavelet characterisations of trace spaces on fractals – Jonsson 1998

Under appropriate assumptions, linear functionals $J : H_\Gamma^{-1/2} \to \mathbb{C}$ (e.g. evaluation of u^s) exhibit expected "superconvergence":

$$|J(\phi) - J(\phi_N)| \lesssim h^{2s+1} \|\phi\|_{H^s_\Gamma}, \qquad h > 0.$$

Example: Cantor dust



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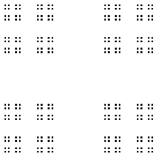

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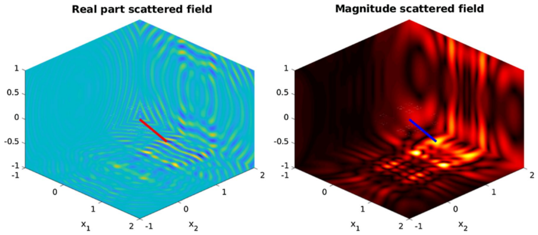

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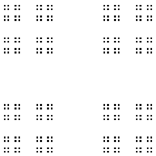

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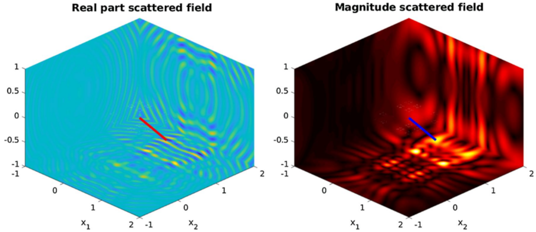

Taking $h = \rho^\ell$, and assuming the best possible regularity, our analysis predicts

$$\|\phi - \phi_N\|_{H_\Gamma^{-1/2}} \lesssim (4\rho)^{-\ell/2}, \qquad |J(\phi) - J(\phi_N)| \lesssim (4\rho)^{-\ell}, \qquad \ell \in \mathbb{N}$$

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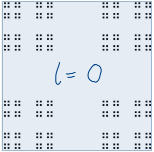

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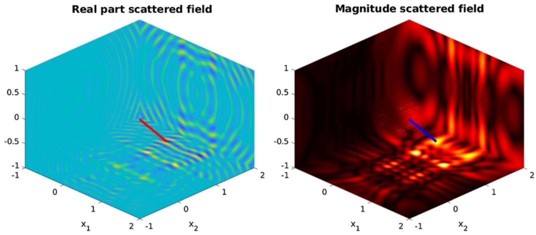

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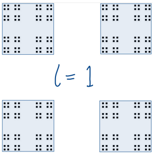

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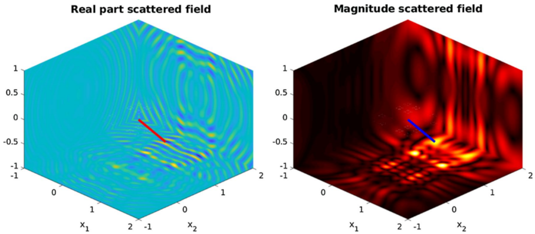

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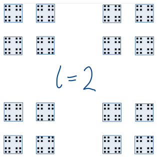

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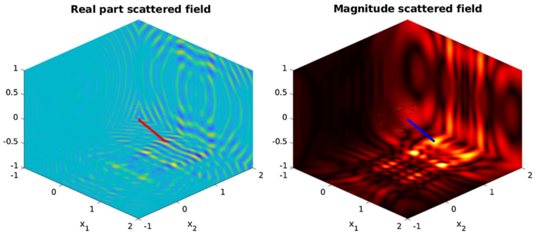

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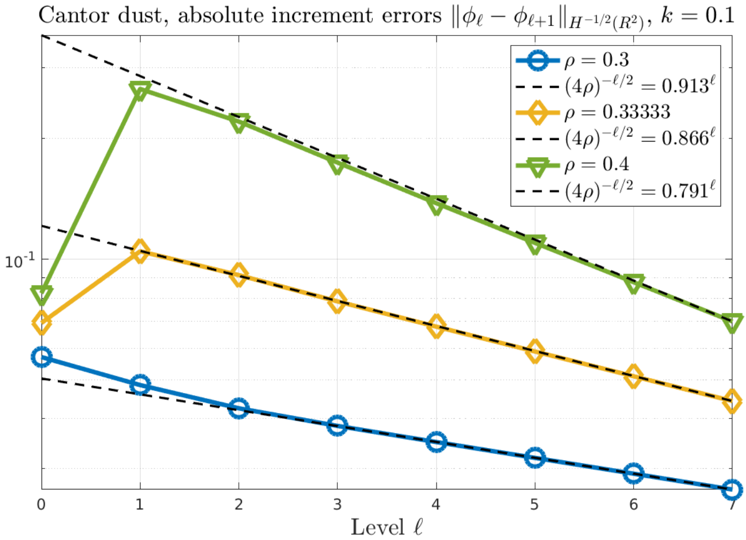

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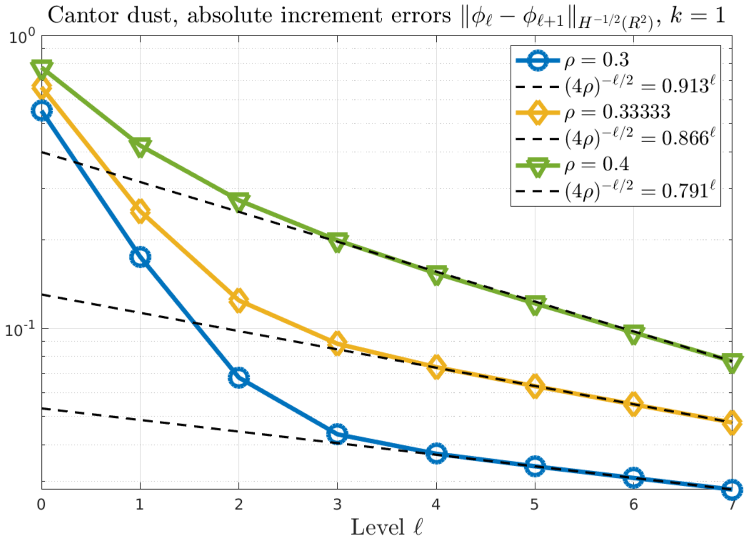

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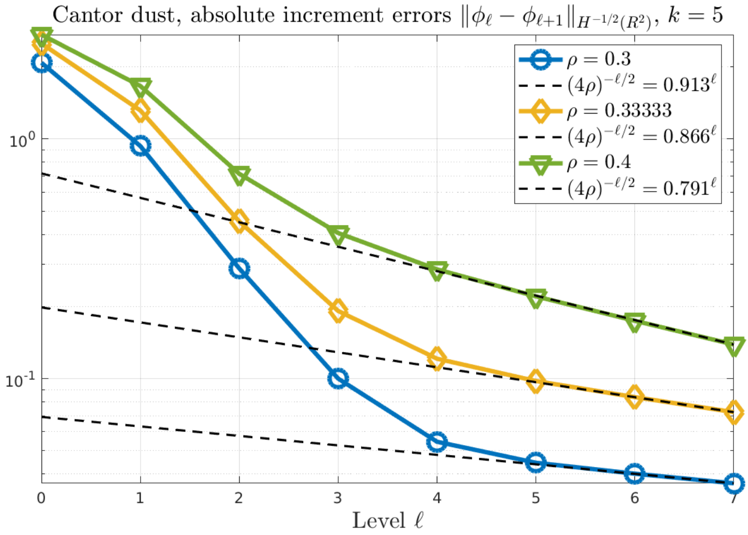

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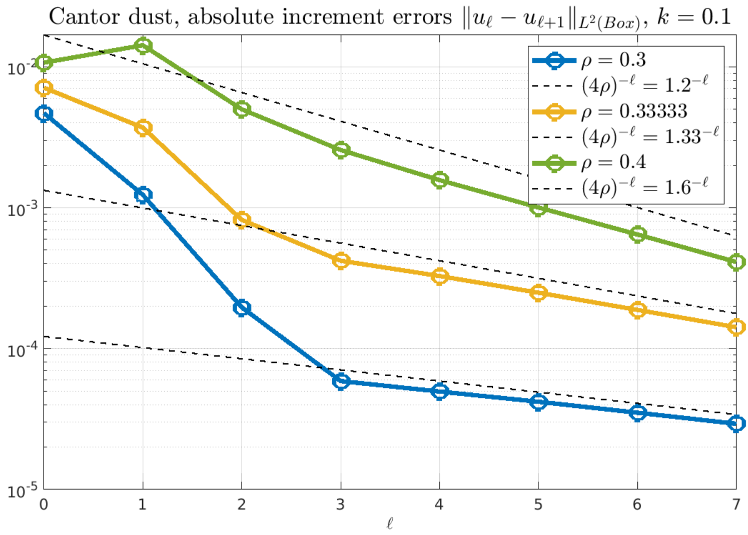

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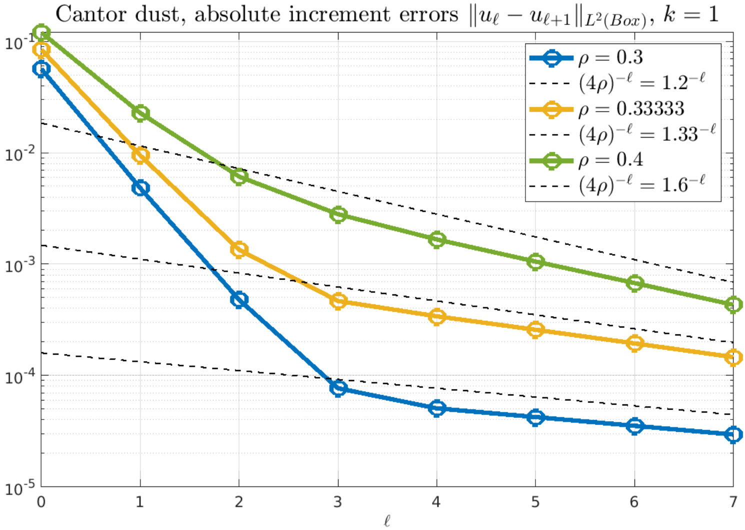

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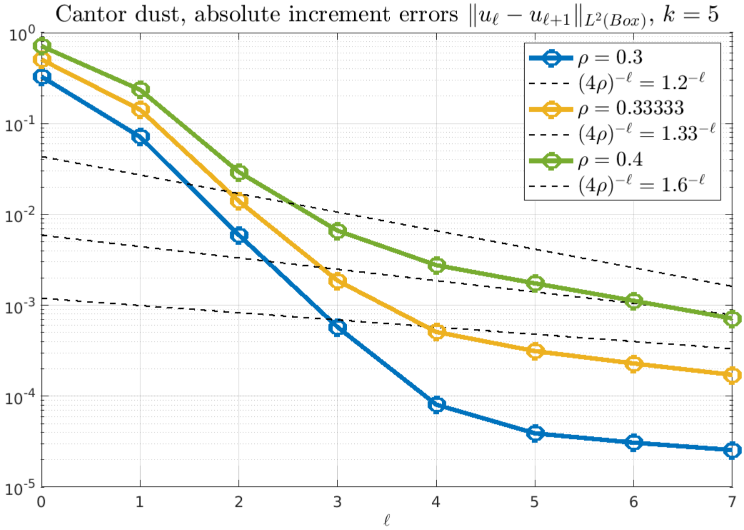

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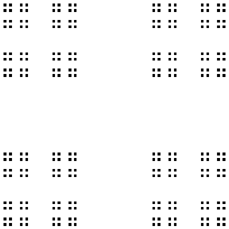
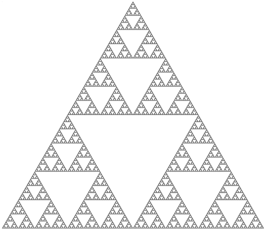
- Currently our analysis requires "disjointness" assumption - future work might include extension to non-disjoint fractals such as the Sierpinski triangle





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- S. N. Chandler-Wilde, D. P. Hewett, A. Moiola, J. Besson, **Boundary element methods for acoustic scattering by fractal screens**, Numer. Math., 147(4), 785-837, 2021
- S. N. Chandler-Wilde, D. P. Hewett, **Well-posed PDE and integral equation formulations for scattering by fractal screens**, SIAM J. Math. Anal., 50(1), 677-717, 2018
- S. N. Chandler-Wilde, D. P. Hewett, **Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens**, Integr. Equat. Operat. Th., 82(3), 423-449, 2015

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Previous work

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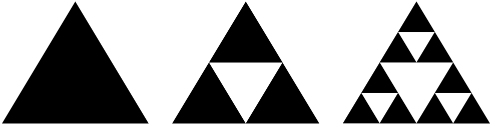

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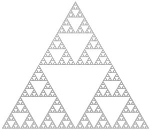

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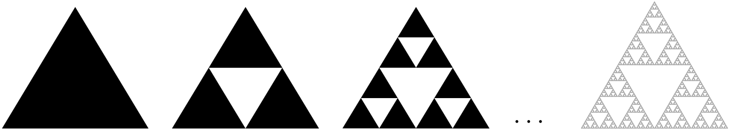

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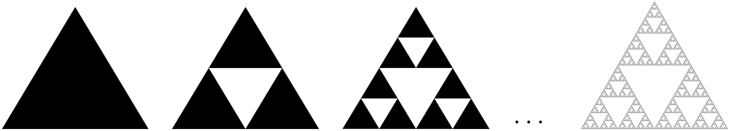

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- Can prove convergence in the joint limit $j \to \infty$, $h \to 0$ using Mosco convergence of function spaces
- However, analysis does not provide convergence rates, and applies only to "thickened" prefractals.
- S. N. Chandler-Wilde, D. P. Hewett, A. Moiola, J. Besson, *Boundary element methods for acoustic scattering by fractal screens*, Numer. Math., 147(4), 785-837, 2021

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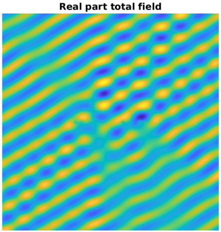

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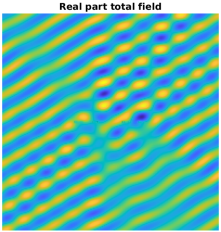

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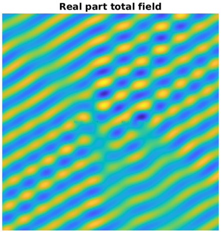

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Note: predicted rates are independent of α, and numerics support this

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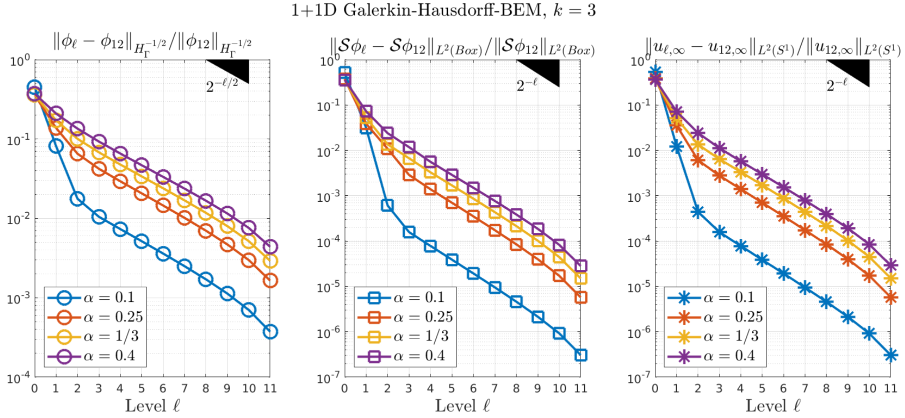

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