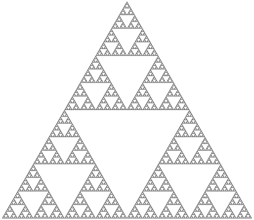
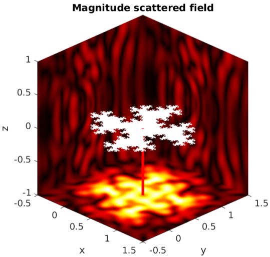
Scattering by fractals: theory and integral equation method computation



Simon Chandler-Wilde

Department of Mathematics and Statistics
University of Reading, UK

University of Reading

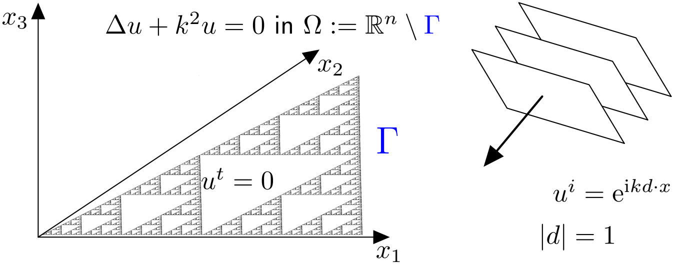


With: Jeanne Besson (ENSTA), António Caetano (Aveiro), Xavier Claeys (Sorbonne), Andrew Gibbs, Dave Hewett (UCL), & Andrea Moiola (Pavia)

CentraleSupélec, Université Paris-Saclay, November 2023

Our focus: sound-soft scattering by very general obstacles

The **obstacle** Γ is some compact subset of $\mathbb{R}^n$, $n = 2, 3$, such that $\Omega := \mathbb{R}^n \setminus \Gamma$ is connected. The **incident**, **scattered**, and **total** fields are u^i, u, and $u^t = u + u^i$, respectively. $k > 0$.



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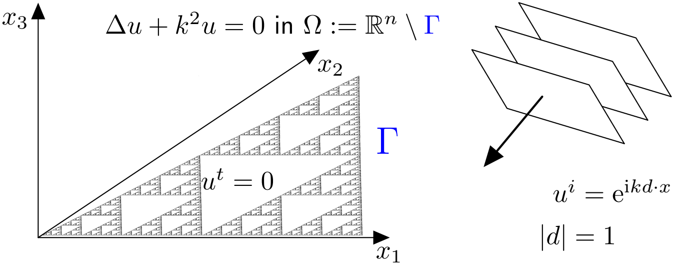

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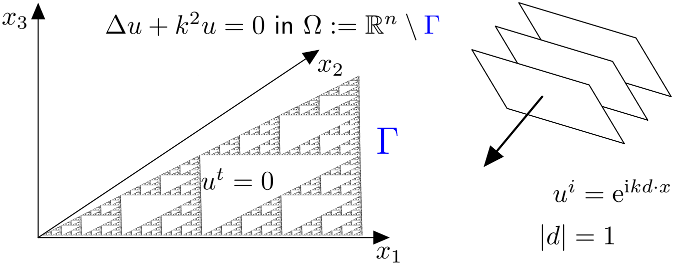

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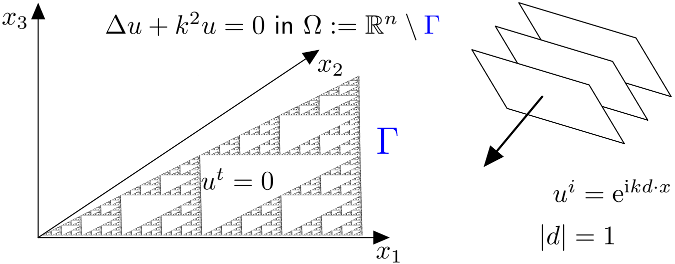

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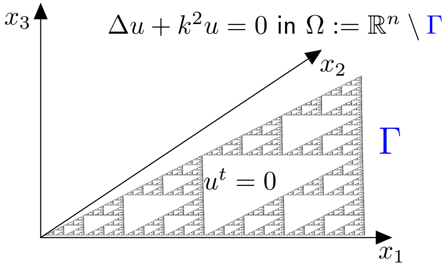

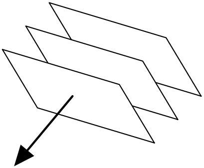

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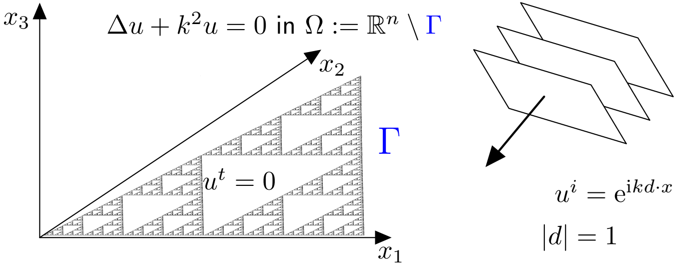

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This scattering problem is well-posed (classical); rewrite as variational problem in $\Omega_R := \{x \in \Omega : |x| < R\}$ with continuous and compactly perturbed coercive sesquilinear form.

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1. Formulating the above scattering problem as a (newish) first kind integral equation

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on Γ, with unknown $\phi \in H_\Gamma^{-1} := \{\psi \in H^{-1}(\mathbb{R}^n) : \mathrm{supp}(\psi) \subset \Gamma\}$.

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2. When Γ is a d-**set**, meaning Γ is uniformly of d-dimensional Hausdorff measure $\mathcal{H}^d$, showing that A can be written as an integral operator $\mathbb{A}$ with respect to $\mathcal{H}^d$, precisely

$$\mathbb{A}\psi(x) = \int_\Gamma \Phi(x,y)\psi(y)\,\mathrm{d}\mathcal{H}^d(y), \quad x \in \Gamma,$$

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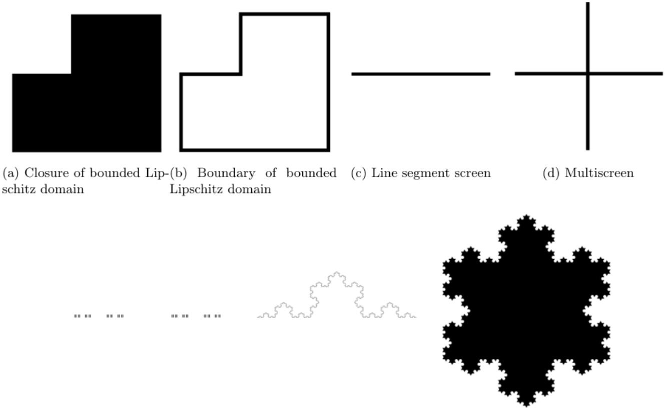
3. When Γ is additionally the **attractor** of an **iterated function system** *of contracting similarities* (an **IFS** for short), proving convergence rates, and providing fully discrete implementation - deferred to next talk by **Dave Hewett** on **Hausdorff-measure integration rules for singular integrals**

What obstacles Γ do our new theories and methods treat?



(a) Closure of bounded Lipschitz domain (b) Boundary of bounded Lipschitz domain (c) Line segment screen (d) Multiscreen

(e) Cantor set screen (f) Koch curve (g) Koch snowflake

Two-dimensional ($n = 2$) examples of d-sets Γ, with: a) $d = 2$; b) $d = 1$; c) $d = 1$; d) $d = 1$; e) $d = \log(2)/\log(3) \approx 0.63$; f) $d = \log(4)/\log(3) \approx 1.26$; g) $d = 2$.

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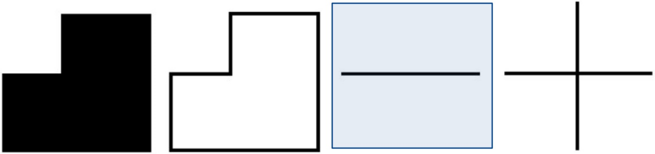

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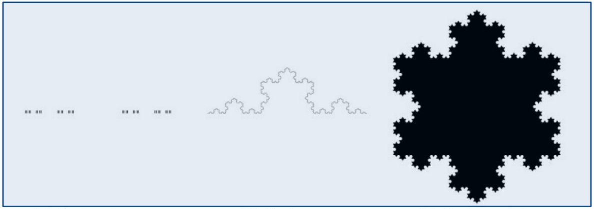

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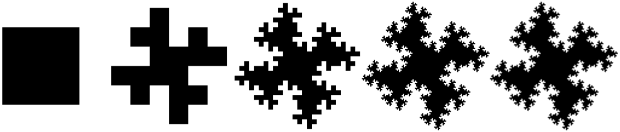
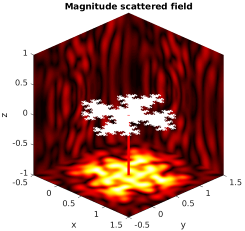
Examples c), e), f), g) are all examples that are attractors of an IFS, for which we have a fully discrete implementation.

Why consider scattering by fractals?





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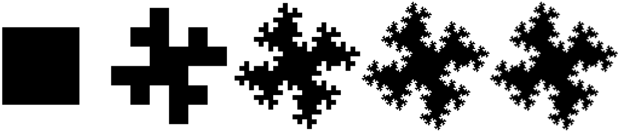

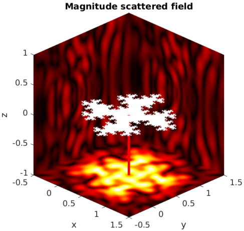

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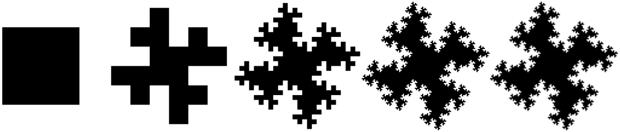

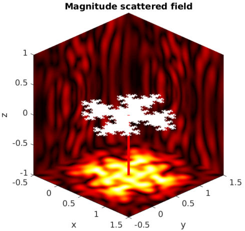

Fractals are an obvious mathematical model for the **multiscale roughness** possessed by many naturally-occuring and man-made scatterers.

They are also a **rich source of mathematical challenges** that are stimulating exciting new research in modelling, function spaces and numerical analysis.

M. V. Berry, "Diffractals", J. Phys. A., 1979 - "*a new regime in wave physics*"

U. Mosco, 2013 - "*introducing fractal constructions into the classic theory of PDEs opens a vast new field of study, both theoretically and numerically*", "*this new field has been only scratched*"

Preliminaries: Sobolev space notation

We need Sobolev spaces **defined on** $\mathbb{R}^n$:

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H^s(\mathbb{R}^n) &:= \Big\{u \in L_2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty\Big\}, \quad s \geq 0, \\
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Preliminaries: Newton potentials

Let $\mathcal{A}\phi$ be the standard acoustic Newton potential, defined for compactly supported $\phi \in L_2(\mathbb{R}^n)$ by

$$\mathcal{A}\phi(x) = \int_{\mathbb{R}^n} \Phi(x,y)\phi(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n,$$

where

$$\Phi(x,y) := \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, \ \ (n=3), \quad := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), \ \ (n=2),$$

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Then $\mathcal{A}$ is continuous as a mapping

$$\mathcal{A} : H^{s-1}_{\mathrm{comp}}(\mathbb{R}^n) \to H^{s+1,\mathrm{loc}}(\mathbb{R}^n), \quad s \in \mathbb{R},$$

where $H^s_{\mathrm{comp}}(\mathbb{R}^n)$ is the space of compactly supported elements of $H^s(\mathbb{R}^n)$, and

$$(\Delta + k^2)\mathcal{A}\phi = \mathcal{A}(\Delta + k^2)\phi = -\phi, \qquad \phi \in H^s_{\mathrm{comp}}(\mathbb{R}^n).$$

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Explicitly for $\phi \in H_\Gamma^{-1} \subset H_{\mathrm{comp}}^{-1}(\mathbb{R}^n)$,

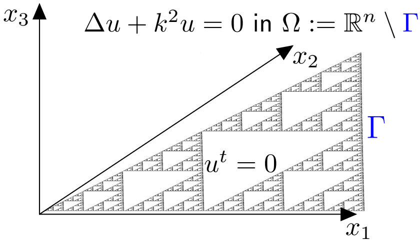
$$\mathcal{A}\phi(x) = \langle \phi, \overline{\sigma\Phi(x,\cdot)}\rangle_{H^{-1}(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}, \qquad x \in \Omega,$$

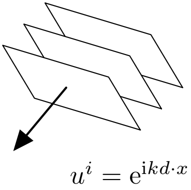
for every

$$\sigma \in C_{0,\Gamma}^\infty := \{\varphi \in C_0^\infty(\mathbb{R}^n) : \varphi = 1 \text{ in a neighbourhood of } \Gamma\},$$

such that $x \notin \mathrm{supp}(\sigma)$.

1. Our integral equation formulation for general Γ





The scattering problem (SP). Find the **scattered field** $u \in H^{1,\mathrm{loc}}(\mathbb{R}^n)$ that satisfies the Helmholtz equation in Ω, the SRC, and that $u^t \in \widetilde{H}^{1,\mathrm{loc}}(\Omega)$.

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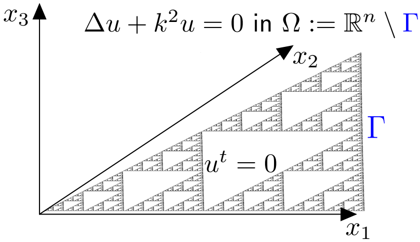

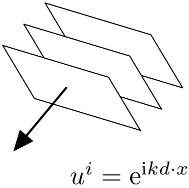

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Let's look for a solution as

$$u = \mathcal{A}\phi \in H^{1,\mathrm{loc}}(\mathbb{R}^n) \quad \text{for some} \quad \phi \in H^{-1}_{\Gamma}.$$

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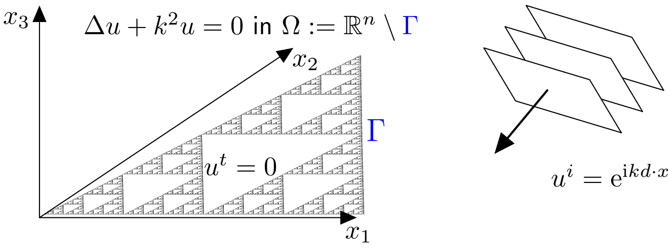

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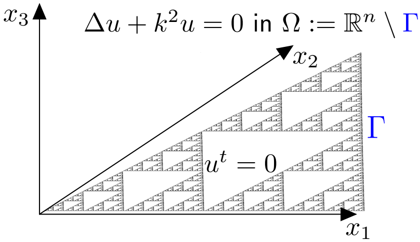

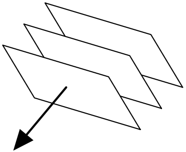

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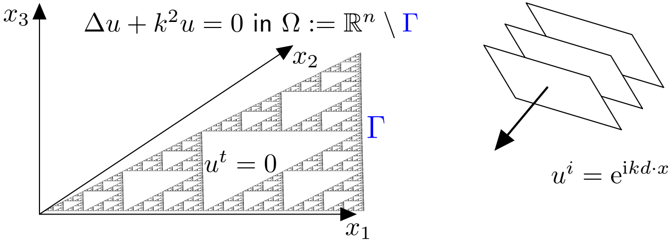

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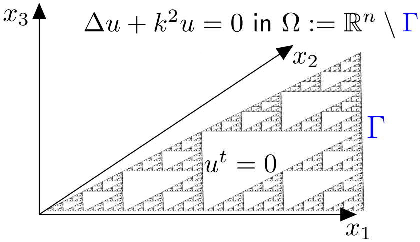

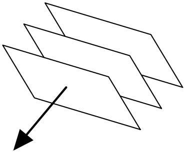

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Proof of main step of 2nd sentence.

$$A = A_k = A_\mathrm{i} + (A_k - A_\mathrm{i}) = \overbrace{A_\mathrm{i}}^{\text{coercive}} + P\overbrace{\sigma(\mathcal{A}_k - \mathcal{A}_\mathrm{i})}^{\text{compact}}.$$

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For all $\psi \in H_\Gamma^{-1}$, since $\mathcal{A}_\mathrm{i} = (1-\Delta)^{-1}$,

$$\begin{aligned}\langle A_\mathrm{i}\psi, \psi\rangle_{\widetilde{H}^1(\Omega)^\perp \times H_\Gamma^{-1}} &= \langle \mathcal{A}_\mathrm{i}\psi, \psi\rangle_{\widetilde{H}^1(\Omega)^\perp \times H_\Gamma^{-1}} \\ &= \langle \mathcal{A}_\mathrm{i}\psi, \psi\rangle_{H^1(\mathbb{R}^n)\times H^{-1}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^{-1}|\widehat{\psi}(\xi)|^2 = \|\psi\|^2_{H_\Gamma^{-1}}.\end{aligned}$$

Preliminaries: Hausdorff measure and dimension, d-sets

For $E \subset \mathbb{R}^n$ and $d \geq 0$,

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This implies that Γ is **uniformly** d-**dimensional** in that

$$\dim_{\mathrm{H}}(\Gamma \cap B_r(x)) = d$$

for every $x \in \Gamma$ and $r > 0$.

Examples of d-sets in two dimensions ($n = 2$)

Given $0 < d \leq n$, a closed set $\Gamma \subset \mathbb{R}^n$ is **a d-set** if there exist $c_1, c_2 > 0$ such that

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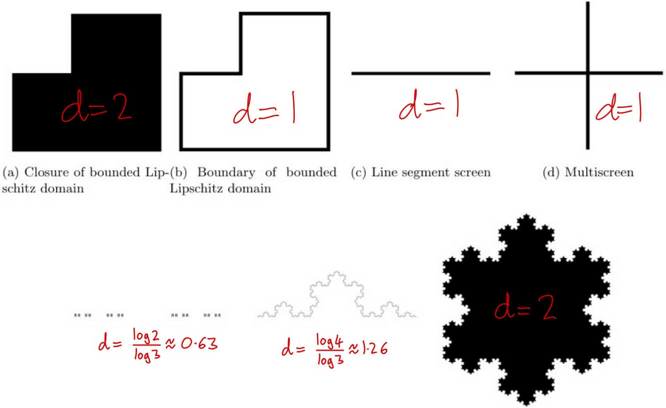

(a) Closure of bounded Lipschitz domain (b) Boundary of bounded Lipschitz domain (c) Line segment screen (d) Multiscreen

(e) Cantor set screen (f) Koch curve (g) Koch snowflake

Trace spaces on d-sets

Let $\Gamma \subset \mathbb{R}^n$ be a d-set and let $\mathbb{L}_2(\Gamma) := \left\{ \Psi : \Gamma \to \mathbb{C} : \int_\Gamma |\Psi|^2 \mathrm{d}\mathcal{H}^d < \infty \right\}$.

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$$\mathbb{H}^{-t}(\Gamma) := (\mathbb{H}^t(\Gamma))', \quad t > 0,$$

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$\mathbb{L}_2(\Gamma)$ is continuously and densely embedded in $\mathbb{H}^{-t}(\Gamma)$ and $\mathrm{tr}_\Gamma^* : \mathbb{H}^{-t}(\Gamma) \to H_\Gamma^{-s}$ is an isometry.

Lemma (Triebel, 2001, Caetano, Hewett, Moiola 2021) For $(n-d)/2 < s < (n-d)/2 + 1$, $\ker(\mathrm{tr}_\Gamma) = \widetilde{H}^s(\Omega)$ where $\Omega := \mathbb{R}^n \setminus \Gamma$, so

$$\mathrm{tr}_\Gamma : \widetilde{H}^s(\Omega)^\perp \to \mathbb{H}^t(\Gamma) \quad \text{and} \quad \mathrm{tr}_\Gamma^* : \mathbb{H}^{-t}(\Gamma) \to H_\Gamma^{-s} = (\widetilde{H}^s(\Omega)^\perp)' \quad \text{are unitary.}$$

2. Our integral equation when Γ is a compact d-set.

Suppose $n-2 < d \leq n$ so $\mathrm{tr}_\Gamma : H^1(\mathbb{R}^n) \to \mathbb{L}_2(\Gamma)$ and $\mathrm{tr}_\Gamma^* : \mathbb{L}_2(\Gamma) \to H_\Gamma^{-1}$ are continuous, and suppose $f \in \mathbb{L}_2(\Gamma)$ so that $\mathrm{tr}_\Gamma^* f \in H_\Gamma^{-1}$.

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$$\mathcal{A}\mathrm{tr}_\Gamma^* f(x) = \langle \mathrm{tr}_\Gamma^* f, \overline{\sigma\Phi(x,\cdot)} \rangle_{H^{-1}(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}, \qquad x \in \Omega,$$

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so that

$$\underbrace{\mathcal{A}\mathrm{tr}_\Gamma^*}_{\mathcal{S}} f(x) = \int_\Gamma \Phi(x,y) f(y) \quad \underbrace{\mathrm{d}\mathcal{H}^d(y),}_{\text{surface measure}} \qquad x \in \Omega.$$

If Γ is boundary of Lipschitz domain then $d = n-1$ and

$$\mathcal{A}\mathrm{tr}_\Gamma^* f = \mathcal{S} f = \text{ standard single-layer potential}$$

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Error Analysis. Recall $A : H^{-1}_\Gamma \to \widetilde{H}^1(\Omega)^\perp = (H^{-1}_\Gamma)'$ is a **compact perturbation** of a **coercive** operator and is **invertible**.

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$$\inf_{\psi_N \in V_N} \|\psi - \psi_N\|_{H_\Gamma^{-1}} = \inf_{f_N \in \mathbb{V}_N} \|f - f_N\|_{\mathbb{H}^{-t_d}(\Gamma)} \to 0 \quad \text{as} \quad N \to \infty,$$

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2. Piecewise constant Galerkin when Γ is a compact d-set.

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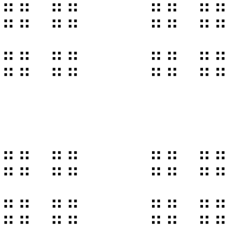
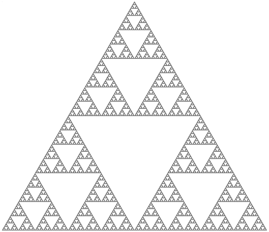


Middle third Cantor dust

$M = 4$, $\rho_m = 1/3$, $d = \log 4 / \log 3$



Sierpinski triangle

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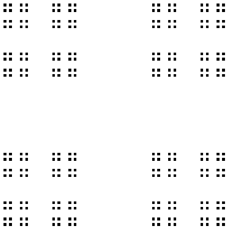

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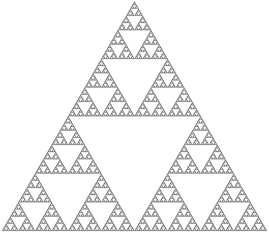

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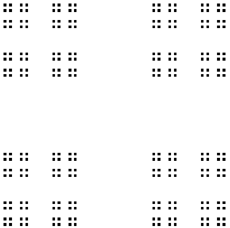

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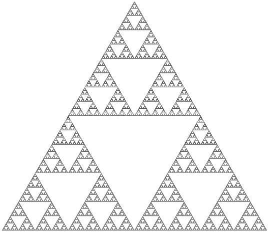

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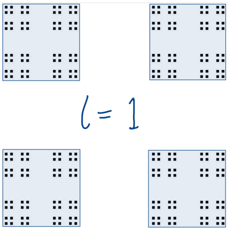

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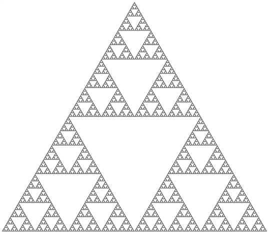

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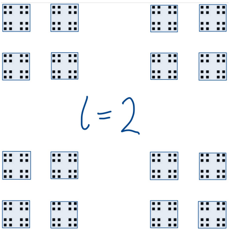

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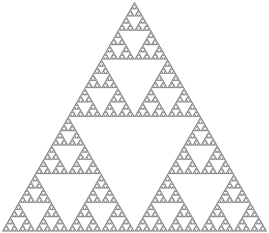

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Under appropriate assumptions, linear functionals $J : H^{-1}_\Gamma \to \mathbb{C}$ (e.g. evaluation of $u = \mathcal{A}\phi(x)$) exhibit expected "superconvergence":

$$|J(\phi) - J(\phi_N)| \lesssim h^{2(s+2)}\|\phi\|_{H^s_\Gamma}, \qquad N \geq N_0.$$

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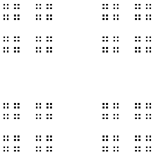
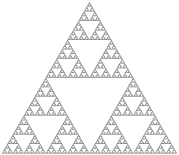
- Currently our analysis requires IFS disjoint - future work might include extension to non-disjoint fractals such as the Sierpinski triangle





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