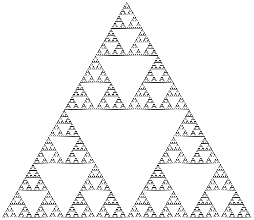
First Kind BIEs and BEM for Acoustic Scattering



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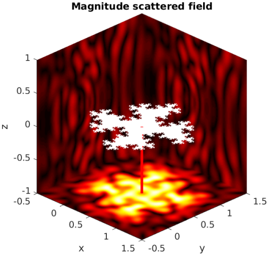

With: Jeanne Besson (ENSTA), António Caetano (Aveiro), Xavier Claeys (Sorbonne), Andrew Gibbs, Dave Hewett (UCL), Andrea Moiola (Pavia), Siavash Sadeghi (Reading)

Rizzo Award Lecture, IABEM 2024, HKUST, Hong Kong SAR, December 2024

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This talk is about what I've worked on throughout my career, namely

solving

$$\Delta u + k^2 u = 0$$

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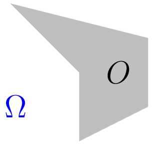

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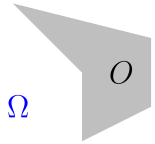

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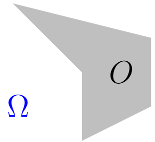
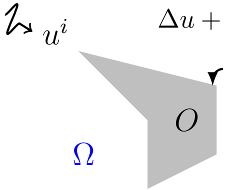


The wave propagation is in $\Omega := \mathbb{R}^n \setminus O$, the complement of and exterior of O, which we assume is **connected**.

This is an old, 19th century problem! The novelty will be results for **general compact** O, including cases where O is **fractal** or has **fractal boundary**.

Given **compact** $O \subset \mathbb{R}^n$ we want to find the **scattered field** u satisfying



$$\Delta u + k^2 u = 0$$

$$u^{\mathrm{t}} := u^{\mathrm{i}} + u = 0$$

u satisfies Sommerfeld radiation condition (SRC),
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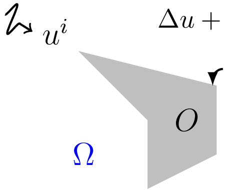

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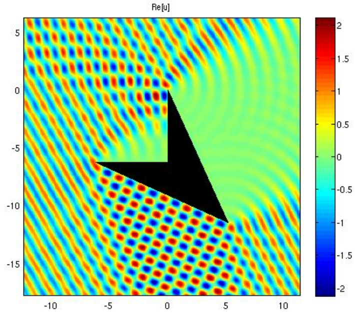
Example 2D Boundary Element Method (BEM) computation when $u^i(x) = \exp(\mathrm{i}kx \cdot \hat{d})$ is a plane wave and O is a polygon, using an asymptotic-numerical hp-BEM and $O(1)$ degrees of freedom as $k \to \infty$ (C-W, Hewett, Langdon, Twigger, 2015).



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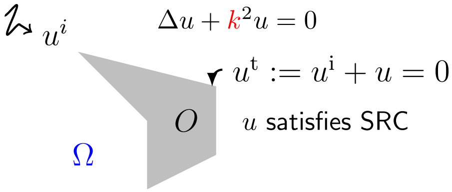
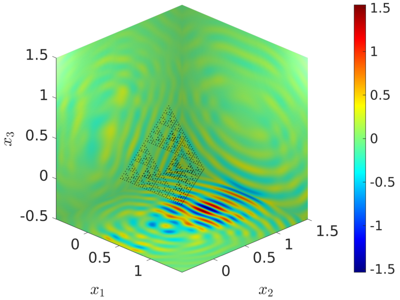


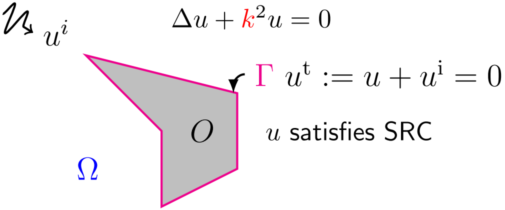
Example 3D BEM computation when $u^i(x) = \exp(\mathrm{i}kx \cdot \hat{d})$ is a plane wave and O is a Sierpinski tetrahedron (Caetano, C-W, Claeys, Gibbs, Hewett, Moiola 2024)



Overview of Talk

1. What is this talk about? (1st kind BIE/BEM for sound-soft acoustic scattering)
2. Standard 1st kind BIEs and (piecewise-constant) BEM for Lipschitz obstacles and screens
3. Our 19th Century roots!
4. A 1st kind IE for general compact obstacles
5. A piecewise-constant Galerkin BEM for (rather) general compact obstacles
6. Numerical examples
7. Conclusion and bibliography

The standard case when O is Lipschitz (e.g., Costabel 1988)



Let $\Gamma := \partial O$ denote the **boundary** of O and let

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ (2D)}, \quad := \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$

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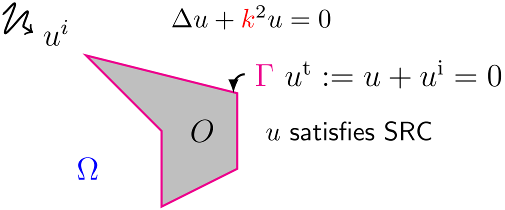

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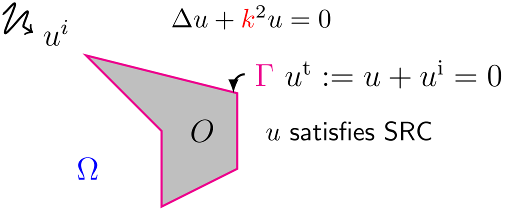

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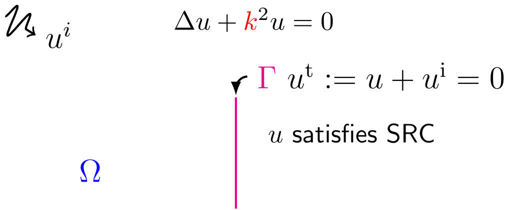
$$-u^{\mathrm{i}}(x) = \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma.$$

...or when $O = \partial O = \Gamma$ is a screen (e.g., Stephan, Wendland 1984)



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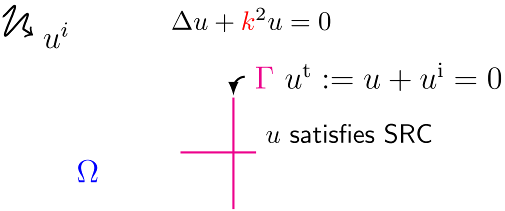

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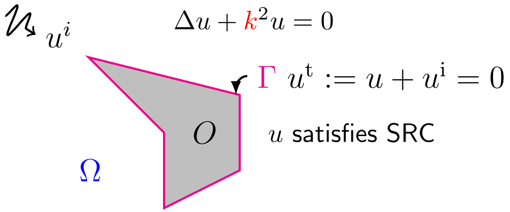

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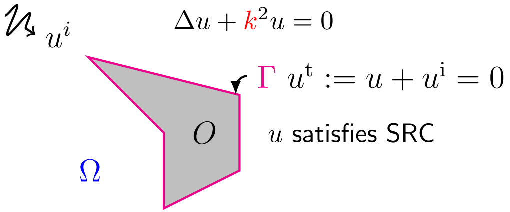

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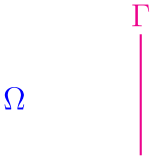
When the obstacle O is Lipschitz this BIE is uniquely solvable (in $H^{-1/2}(\Gamma)$) if and only if k is not an **irregular frequency**, i.e., k^2 is not a Dirichlet eigenvalue of $-\Delta$ in

$$\Omega_- := {\rm int}(O) = O \setminus \Gamma.$$

Piecewise-constant Galerkin BEM



Galerkin method for the **BIE**:

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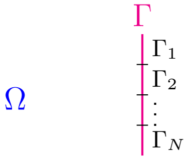

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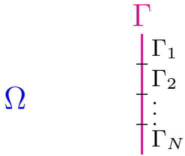

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Determine $\varphi_1, \dots \varphi_N$ by solving the Galerkin equations

$$\sum_{j=1}^{N} \int_{\Gamma_i} \int_{\Gamma_j} \Phi(x,y)\,ds(x)ds(y)\,\varphi_j = -\int_{\Gamma_i} u^{\mathrm{i}}(x)\,ds(x), \qquad i = 1 \dots N.$$

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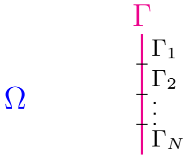

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Approximate $u(x)$ by

$$u_N(x) := \sum_{j=1}^{N} \int_{\Gamma_j} \Phi(x,y)\,ds(y)\,\varphi_j, \qquad x \in \Omega.$$

These are very old ideas!

For the **BIE for a flat screen** (equivalently an **aperture** in an infinite screen), see J. W. S. Rayleigh, *Theory of Sound, Vol. 2,* London: Macmillan, 1878

If $P\cos(nt+\epsilon)$ denote the value of $d\phi/dx$ at the various points of the area (S) of the aperture, the condition for determining P and ϵ is by (6) § 278,

$$-\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS = \cos nt \quad\text{.........(2)},$$

where r denotes the distance between the element dS and any fixed point in the aperture. When P and ϵ are known, the complete value of ϕ for any point on the positive side of the screen is given by

$$\phi = -\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS \quad\text{...........................(3)},$$

and for any point on the negative side by

$$\phi = +\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS + 2\cos nt\cos kx \quad\text{......(4)}.$$

The expression of P and ϵ for a finite aperture, even if of circular form, is probably beyond the power of known methods; but in the

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For the first **piecewise-constant BEM calculations in 3D for a flat screen** for $k = 0$ see ... J. C. Maxwell, *The Electrical Researches of the Honourable Henry Cavendish FRS*, Cambridge: University Press, 1879

NOTE 22, ART. 283.

Electric Capacity of a Square.

I am not aware of any method by which the capacity of a square can be found exactly. I have therefore endeavoured to find an approximate value by dividing the square into 36 equal squares and calculating the charge of each so as to make the potential at the middle of each square equal to unity.

The potential at the middle of a square whose side is 1 and whose charge is 1, distributed with uniform density, is

$$4 \log (1 + \sqrt{2}) = 3{\cdot}52549.$$

In calculating the potential at the middle of any of the small squares which do not touch the sides of the great square I have used this formula, but for those which touch a side I have supposed the value to be 3·1583, and for a corner square 2·9247.

If the 36 squares are arranged as in the margin, and if the charges of the corner squares be taken for unity, the charges will be as follows:

A	B	C	C	B	A
B	D	E	E	D	B
C	E	F	F	E	C
C	E	F	F	E	C
B	D	E	E	D	B
A	B	C	C	B	A

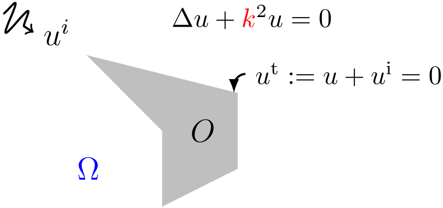
A	B	C	D	E	F
1·000	·599	·562	·265	·210	·201

and the capacity of a square whose side is 1 will be 0·3607.

The ratio of the capacity of a square to that of a globe whose diameter is equal to a side of the square is therefore 0·7214.

Now for the new stuff!

An IE for general compact obstacles: Case $\mathrm{Im}\, k > 0$



Recall O is compact and $\Omega := \mathbb{R}^n \setminus O$ is connected, and assume that

$$u^{\mathrm{i}} \in H^1(\mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n) : \nabla v \in L^2(\mathbb{R}^n)\}.$$

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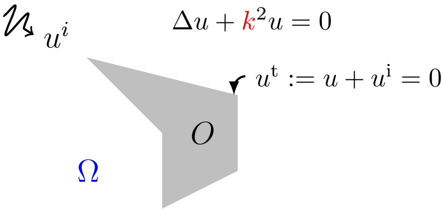

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The scattering problem. Find the scattered field $u \in H^1(\Omega)$ that satisfies the Helmholtz equation in Ω and that $u^{\mathrm{t}} = 0$ on $\partial\Omega = \partial\Gamma$ in the sense that $u^{\mathrm{t}} \in H_0^1(\Omega)$.

Here

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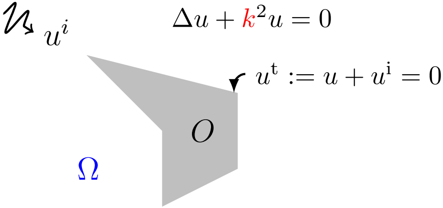

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It is a standard PDE result (via a variational formulation and Lax-Milgram) that this problem is well-posed.

A little more Sobolev space notation: (McLean 2000)

We've just introduced

$$H^1(\mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n) : \nabla v \in L^2(\mathbb{R}^n)\}$$

and spaces of functions **defined on** Ω:

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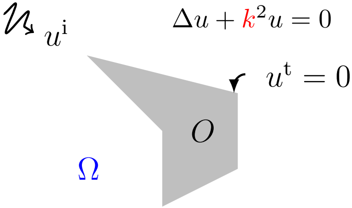
N.B. $\widetilde{H}^1(\Omega)$ and $H_0^1(\Omega)$ are almost the same space: precisely, restriction to Ω is an isometric isomorphism

$$|_\Omega : \widetilde{H}^1(\Omega) \to H_0^1(\Omega)$$

whose inverse is extension by zero.

IE for general compact O: Case $\mathrm{Im}\, k > 0$ (Caetano et al 2024)



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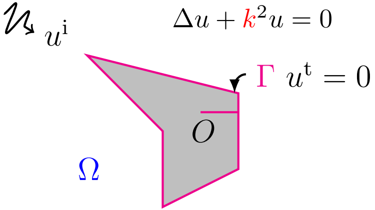

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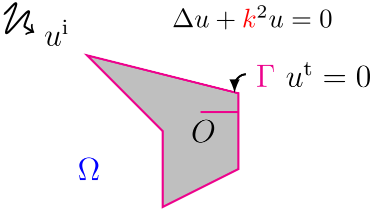

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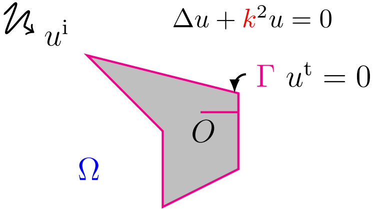

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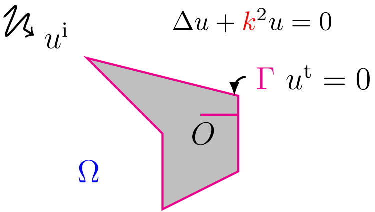

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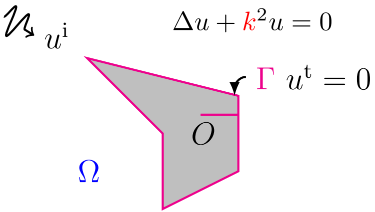

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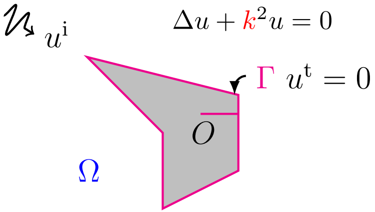

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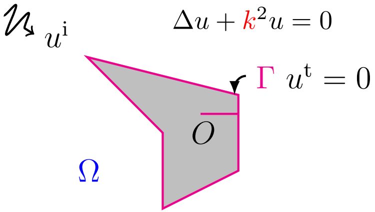

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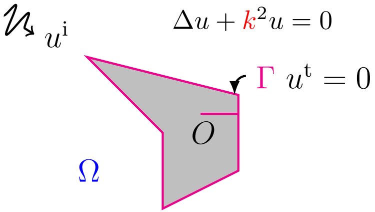

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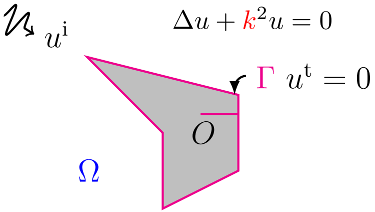

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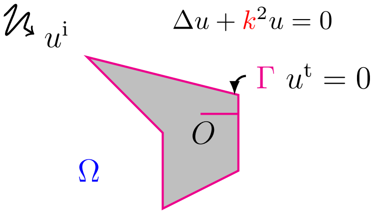

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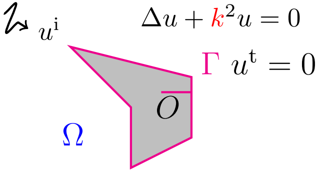

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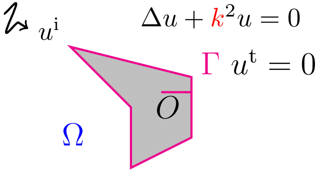

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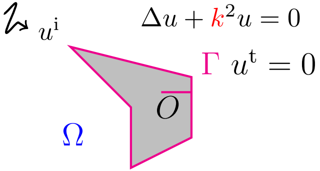

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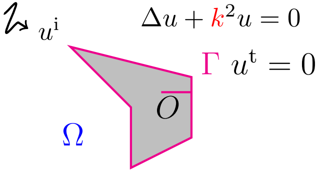

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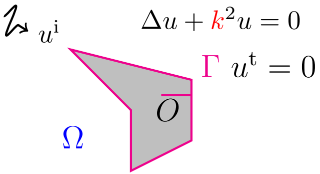

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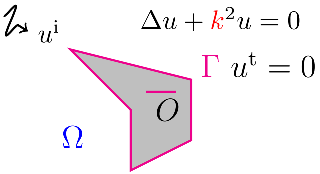

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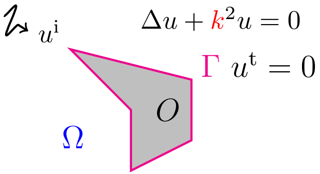

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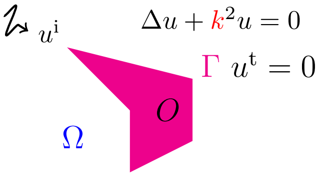

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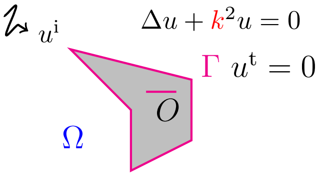

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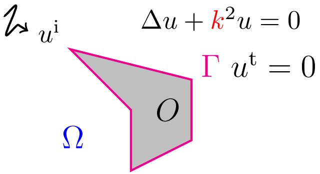

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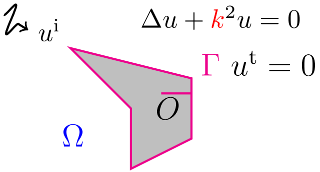
Lemma (Caetano et al 2024)

If $\Delta u^{\mathrm{i}} + k^2 u^{\mathrm{i}} = 0$ in a neighbourhood of O and $\mathbf{S}_k$ is invertible, then $\phi = \mathbf{S}_k^{-1} g \in H_{\partial\Omega}^{-1}$.

Overview of Talk

Aim: Solve $\mathbf{S}_k \phi = g$ by piecewise-constant Galerkin BEM, for which we need **a notion of integration on Γ.**



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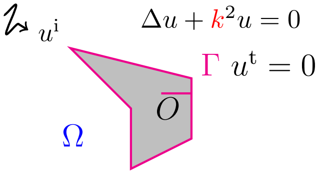

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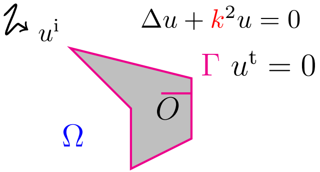

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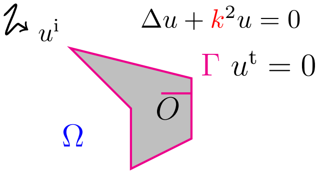

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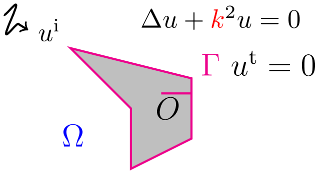

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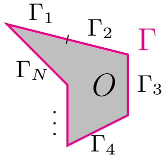

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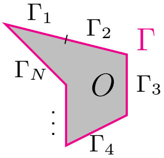
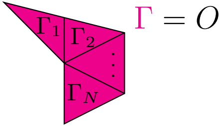


2. Domain IE Case. $\Gamma = O$ is Lipschitz, μ is **(n-dimensional) Lebesgue measure,** μ_L



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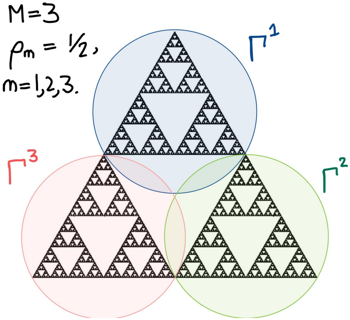

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$M=3$
$\rho_m = 1/2$,
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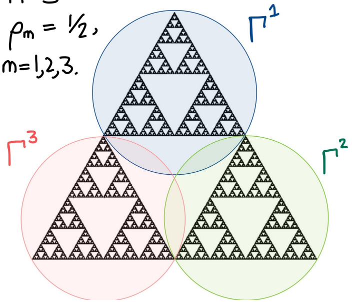

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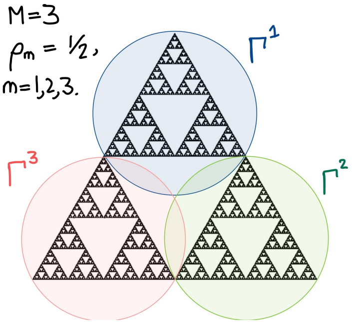

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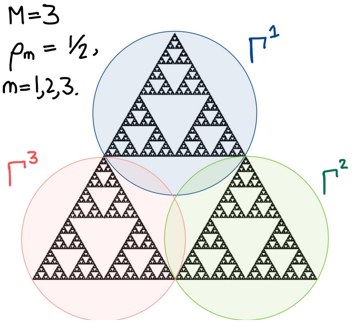

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This is simple! For this example just take $\mu(\Gamma) = 1$, $\mu(\Gamma^m) = 1/3$, $m = 1, 2, 3$, etc.

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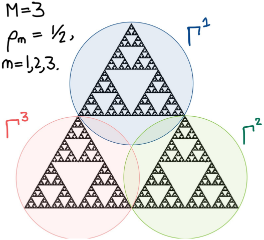

$$d = \log 3/\log 2 \approx 1.58$$

As long as

$$n - 2 < d \leq n,$$

$\mu = c\mu^d$ works, where $c > 0$ and μ^d is d-**dimensional Hausdorff measure.**
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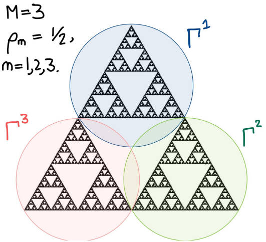

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$\ell = 1$, $N = 3$, $\Gamma_m = \Gamma^m$, $m = 1, 2, 3$, $h_N = \operatorname{diam}(\Gamma)/3$.

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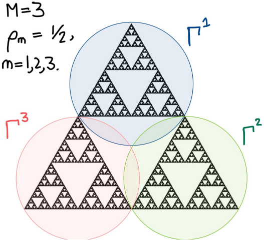

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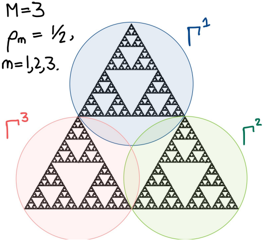

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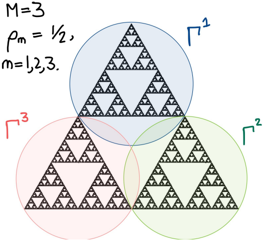

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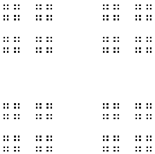

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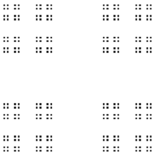

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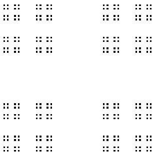

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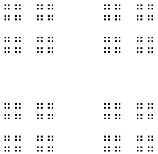

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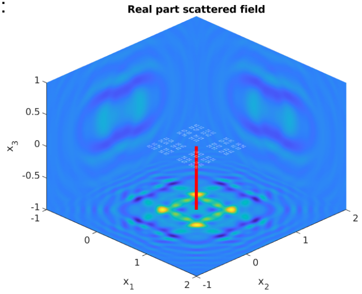

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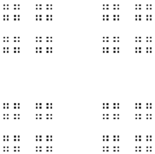

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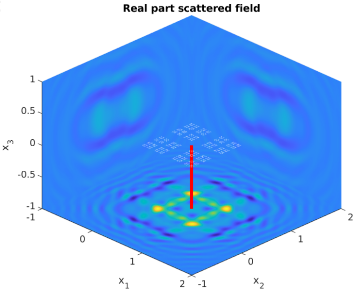

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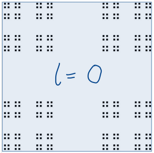

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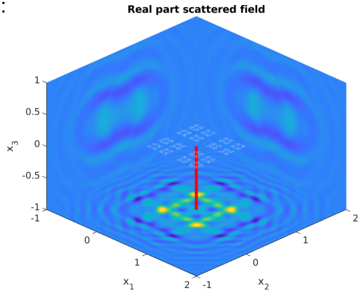

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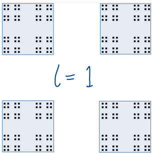

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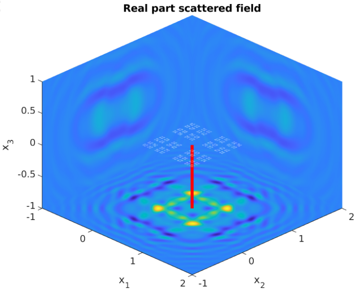

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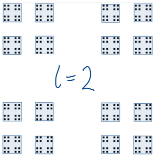

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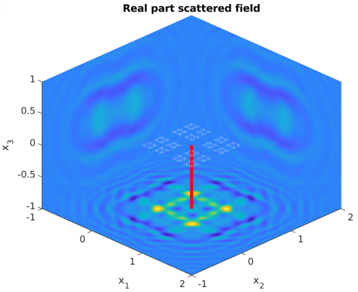

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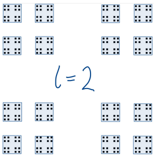

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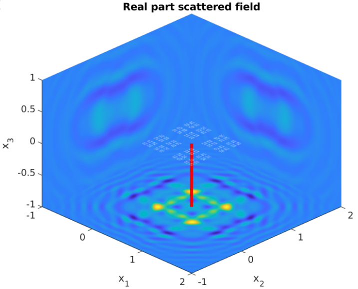
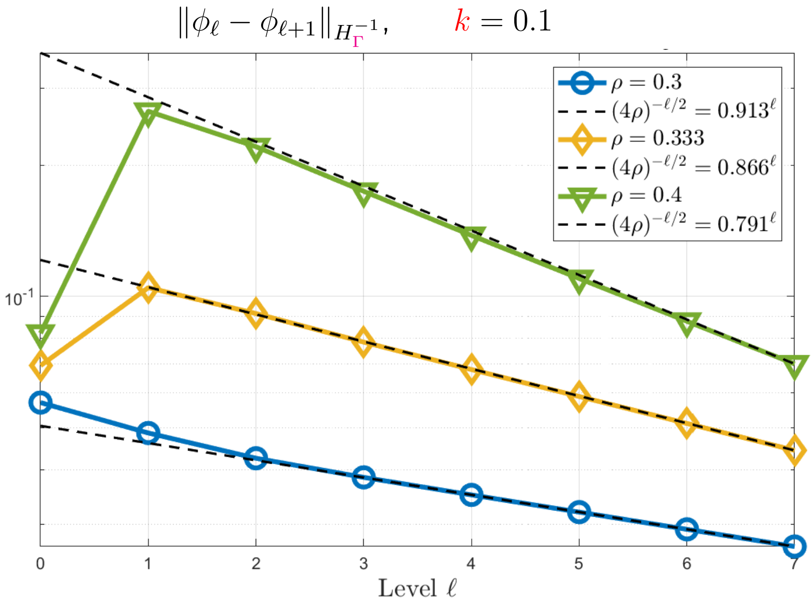


We take $N = 4^\ell$ elements so that $h_N = \rho^\ell \operatorname{diam}(\Gamma)$. Assuming best possible solution regularity, a **wavelet-based** best-approximation analysis (using Jonsson 1998) gives

$$\|\phi - \phi_N\|_{H^{-1}_{\Gamma}} \lesssim h_N^{(d-1)/2} \approx (4\rho)^{-\ell/2}, \qquad |u(x) - u_N(x)| \lesssim h_N^{d-1} \approx (4\rho)^{-\ell}.$$

Numerical results – 3D – scattering by Cantor dust



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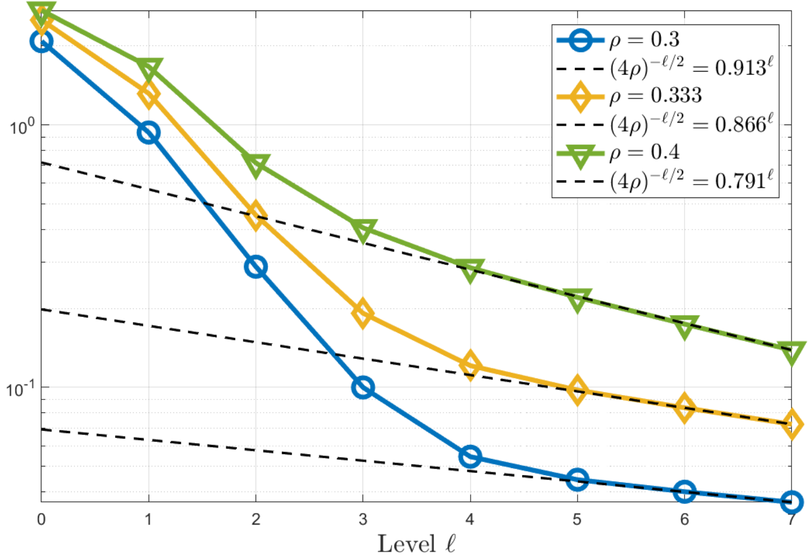
$$\|\phi_\ell - \phi_{\ell+1}\|_{H_\Gamma^{-1}}, \qquad k = 5$$



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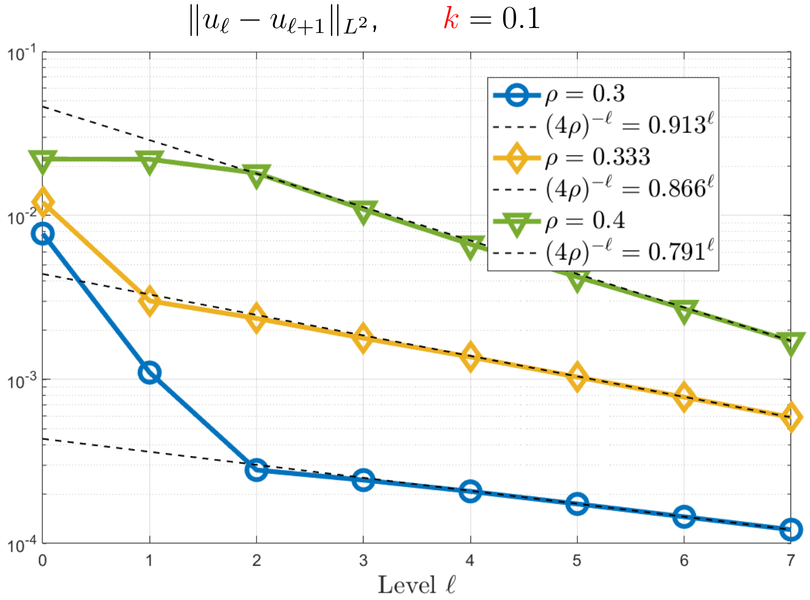

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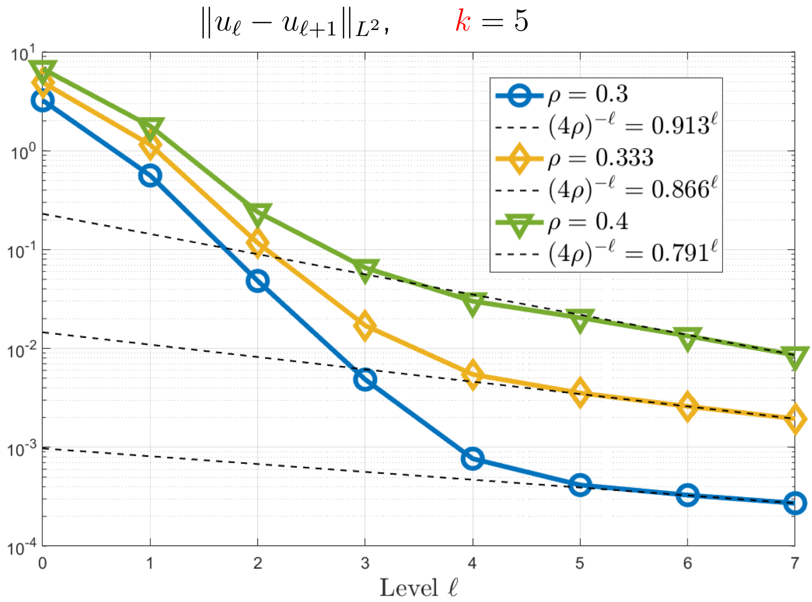

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- Not discussed today is wavenumber dependence (recent work with Sadeghi), in which we show that

$$\|\mathbf{S}_k^{-1}\| \lesssim \begin{cases} k, & \text{for } k \geq k_0 \text{ if } \Gamma \text{ star-shaped,} \\ k^{2n+2+\delta}, & \text{for } k \in [k_0, \infty) \setminus E \text{ in general,} \end{cases}$$

for every $\delta > 0$ and some $E \subset [k_0, \infty)$ of arbitrarily small Lebesgue measure.

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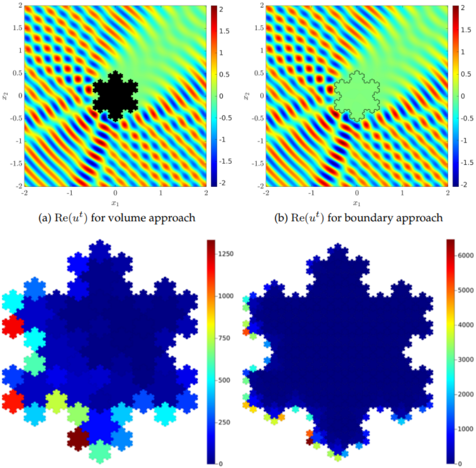
A 2D example: O is a Koch snowflake

Choice 1: $\Gamma = \partial O$, the BIE choice, so $\Omega_- = \mathrm{int}(O)$.

Choice 2: $\Gamma = O$, so $\Omega_- = \emptyset$, and $\mathbf{S}_k$ invertible for all $k > 0$.



(a) $\mathrm{Re}(u^t)$ for volume approach

(b) $\mathrm{Re}(u^t)$ for boundary approach

(c) $|\phi_N|$ for volume approach, $h = 0.22$

(d) $|\phi_N|$ for volume approach, $h = 0.074$