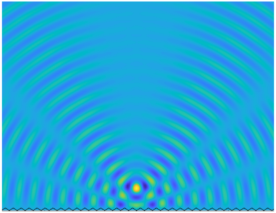
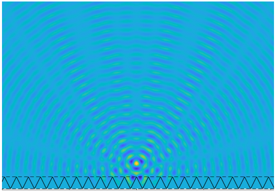
Integral equations and boundary element methods for rough surface scattering (RSS)



Simon Chandler-Wilde

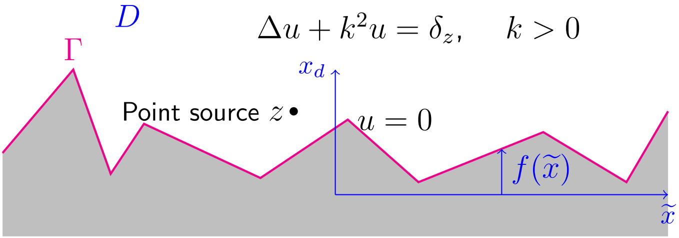
Department of Mathematics
and Statistics
University of Reading, UK

University of Reading

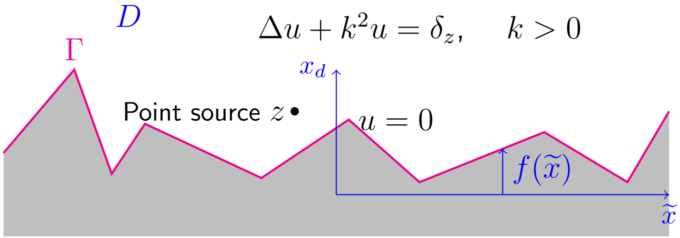


With: Martin Averseng & Euan Spence (Bath, UK)

INI Computational Methods for Multiple Scattering Workshop, April 2023



Many interesting **computational and numerical analysis challenges**!



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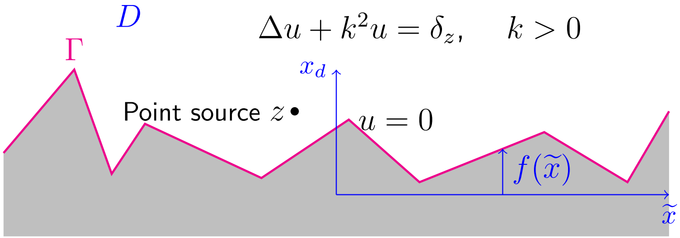

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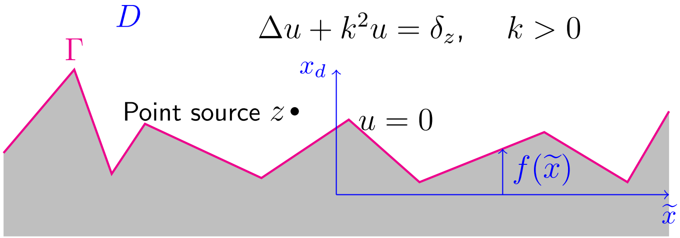

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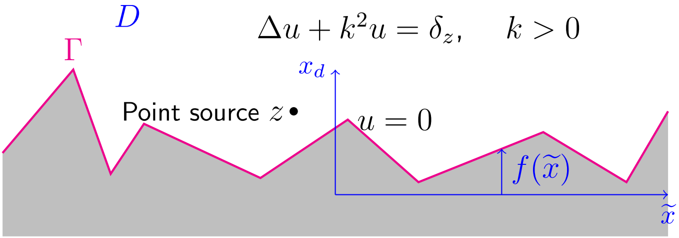

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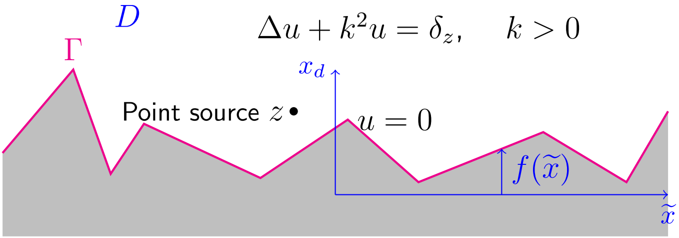

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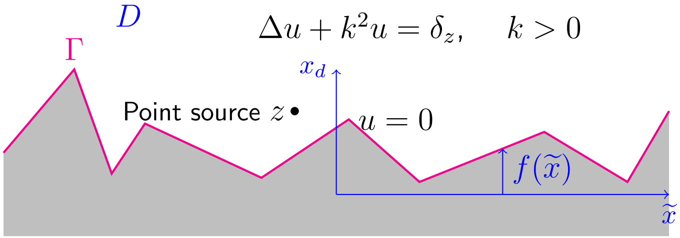

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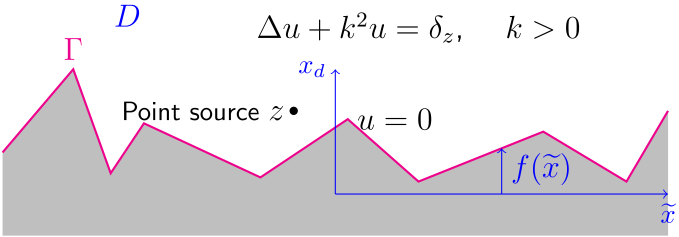

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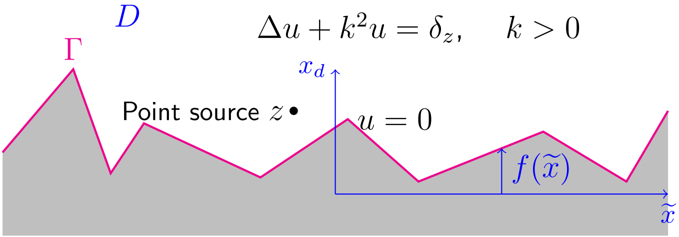

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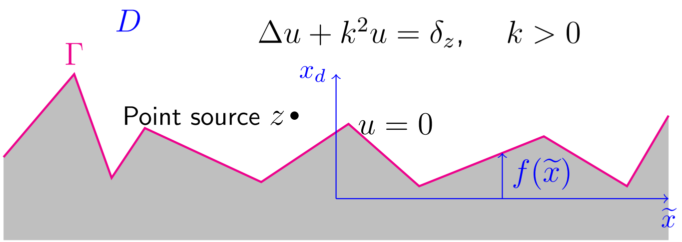

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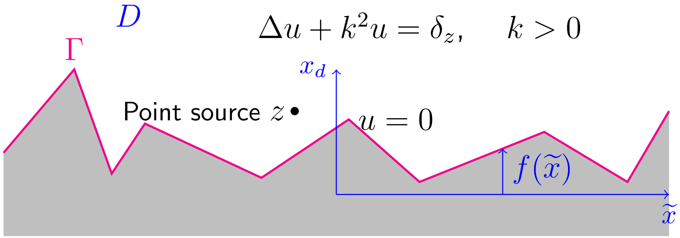

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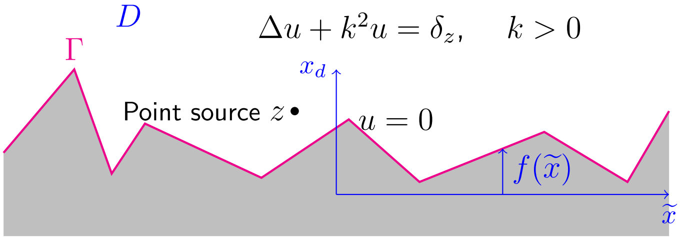

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Tools for convergence of Galerkin methods and GMRES

Suppose that H is a complex Hilbert space with norm $\|u\| = \sqrt{(u,u)}$, e.g.

$$H = L^2(\Gamma), \quad (u,v) = \int_\Gamma u\bar{v}\,\mathrm{d}s, \quad \|u\|^2 = \int_\Gamma |u|^2\,\mathrm{d}s.$$

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Céa's Lemma. Let $H_N \subset H$ be a closed subspace. Then, $\forall g \in H$, $\exists$ a unique *Galerkin approximation* $u_N \in H_N$ to $u := A^{-1}g$, defined by

$$(Au_N, v_N) = (g, v), \quad \forall v_N \in H_N,$$

and

$$\|u - u_N\| \leq \frac{\|A\|}{\gamma} \inf_{v_N \in H_N} \|u - v_N\|.$$

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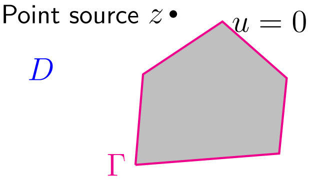
Theorem *(corollary of field of values estimate in Beckermann et al. 2006)*. Let r_m be the residual after m steps of GMRES applied to (X). Then

$$\frac{\|r_m\|_2}{\|r_0\|_2} \leq \varepsilon \quad \text{provided} \quad m \geq \frac{3\sqrt{3}}{4} \frac{\|A\|}{\gamma} \operatorname{cond}(M) \log\left(\frac{8}{\varepsilon}\right),$$

where $M = [(\varphi_n, \varphi_m)]$ is the mass matrix.

Integral equation methods: bounded obstacle case



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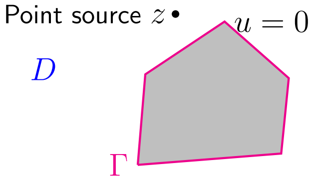

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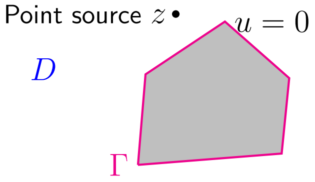

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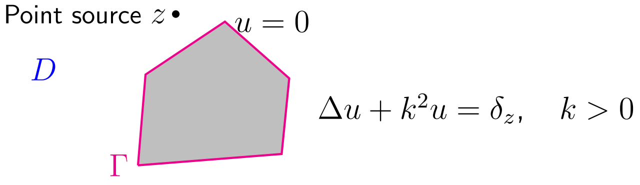

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$$K'\varphi(x) := \int_\Gamma \partial_{n(x)}\Phi(x,y)\varphi(y)\,ds(y), \quad S\varphi(x) := \int_\Gamma \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \Gamma.$$

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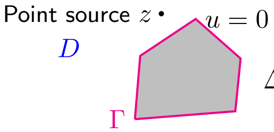

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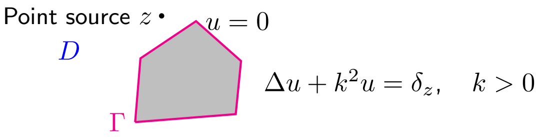

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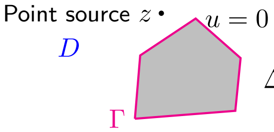

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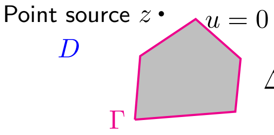

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- A is **uniformly-in-k coercive**, i.e., for all $k_0 > 0$ there exists $\gamma > 0$ such that

$$|(A\varphi, \varphi)| \geq \gamma \|\varphi\|^2, \quad \varphi \in L^2(\Gamma), \quad k \geq k_0,$$

  if Γ is **smooth and uniformly convex** (Spence, Kamotski, Smyshlyaev 2016)

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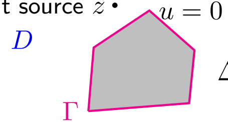

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$$A\partial_n u = g := \partial_n u^i - \mathrm{i}k\gamma u^i, \quad \text{and} \quad A := \tfrac{1}{2}I + K' - \mathrm{i}kS.$$

- $\|A^{-1}\| = O(1)$ as $k \to \infty$ if Γ is star-shaped or smooth and non-trapping (C-W & Monk 2008, Baskin, Spence, Wunsch 2016)
- $\|A\| = O(k^{1/2})$ as $k \to \infty$ if Γ is star-shaped and 2D, or smooth and non-trapping (C-W et al. 2009, Baskin, Spence, Wunsch 2016)
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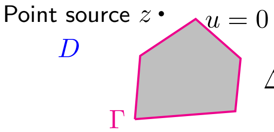
  $$|(A\varphi, \varphi)| \geq \gamma \|\varphi\|^2, \quad \varphi \in L^2(\Gamma), \quad k \geq k_0,$$

  if Γ is **smooth and uniformly convex** (Spence, Kamotski, Smyshlyaev 2016)
- **BUT** A is not even compactly perturbed coercive for general Lipschitz Γ, or even for general star-shaped polyhedra in 3D (C-W & Spence 2022a)

Integral equation methods: bounded obstacle case



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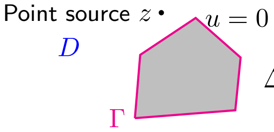
- **BUT** A is not even compactly perturbed coercive for general Lipschitz Γ, or even for general star-shaped polyhedra in 3D (C-W & Spence 2022a) **AND** there is no numerical method provably convergent for every polyhedron Γ (**open problem)**.

Coercive formulations: bounded obstacle case



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Taking a linear combination of Dirichlet (γ) and Neumann (∂_n) traces, we obtain the standard 2nd kind integral equation

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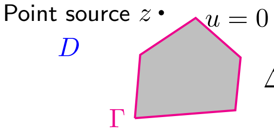

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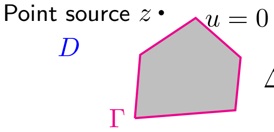

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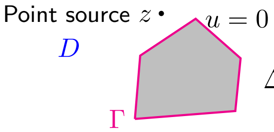

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Theorem *(C-W & Spence 2022b)*. If Z is continuous and $Z \cdot n \geq c > 0$ on Γ, then $A_Z = A_0 + K$ where A_0 is coercive and K is compact, so that all Galerkin methods for $A_Z \partial_n u = g_Z$ are convergent, provided A_Z is injective.

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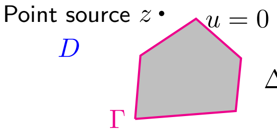

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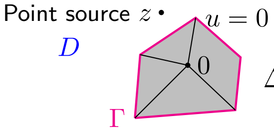
Sadly injectivity of A_Z not yet clear in general (**open problem**).

Coercive formulations: bounded obstacle case



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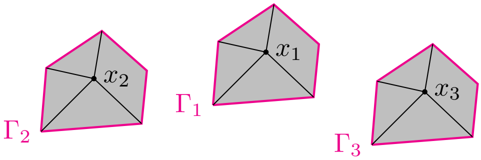
Theorem *(Spence, C-W, Graham, Smyshlyaev 2011)*. If Γ is **star-shaped** with respect to 0,

$$Z(x) := x, \quad \alpha(x) := k|x| + \mathrm{i}(d-1)/2, \quad x \cdot n \geq c > 0,$$

on Γ, then A_Z is **uniformly-in-k coercive** with coercivity constant $\gamma = c/2$, so that all Galerkin methods for $A_Z \partial_n u = g_Z$ are convergent.

Multiple scattering formulation



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Corollary *(Gibbs, C-W, Langdon, Moiola 2021)*. If each component Γ_j of Γ is star-shaped, and, on Γ_j,

$$Z(x) := x - x_j, \quad \alpha(x) := k|x - x_j| + \mathrm{i}(d-1)/2, \quad (x - x_j) \cdot n \geq c > 0,$$

then $A_Z = A_0 + K$ with A_0 coercive and K compact, **and** A_Z is injective, so that all Galerkin methods for $A_Z \partial_n u = g_Z$ are convergent.

Our typical RSS problem

Suppose $d = 2$ or 3, $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is bounded and Lipschitz continuous, precisely

$$0 < f_- \leq f(\widetilde{x}) \leq f_+ \quad \text{and} \quad |f(\widetilde{x}) - f(\widetilde{y})| \leq L|\widetilde{x} - \widetilde{y}|, \quad \widetilde{x}, \widetilde{y} \in \mathbb{R}^{d-1}.$$

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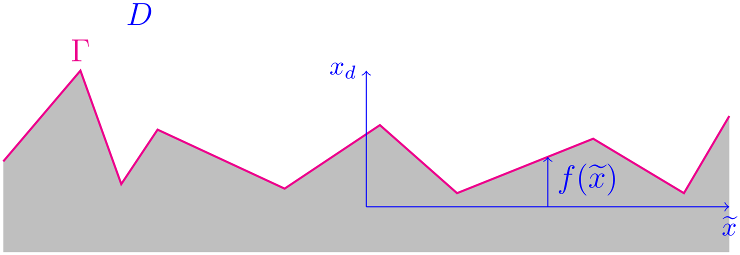

$$D := \{(\widetilde{x}, x_d) : x_d > f(\widetilde{x}),\ \widetilde{x} \in \mathbb{R}^{d-1}\} \subset \mathbb{R}^d, \quad \Gamma := \partial D = \{(\widetilde{x}, f(\widetilde{x})) : \widetilde{x} \in \mathbb{R}^{d-1}\}.$$

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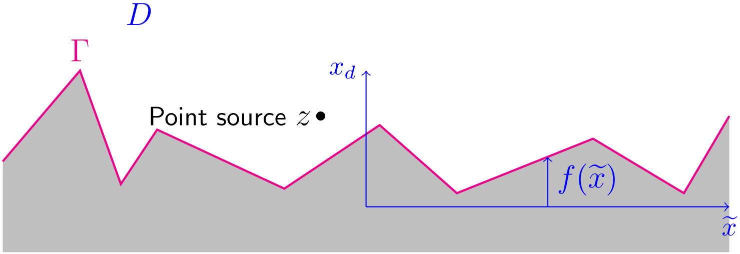

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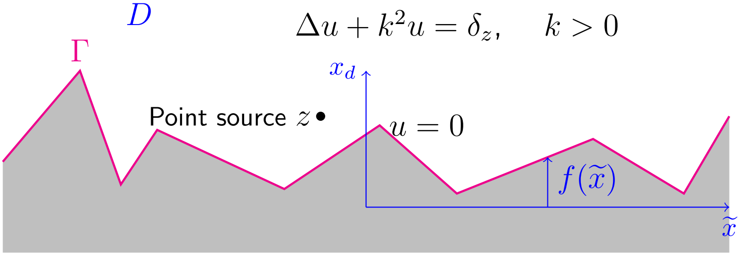

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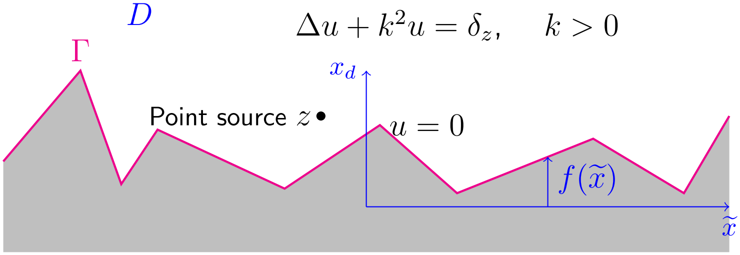

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Key feature: Γ unbounded (in the horizontal directions).

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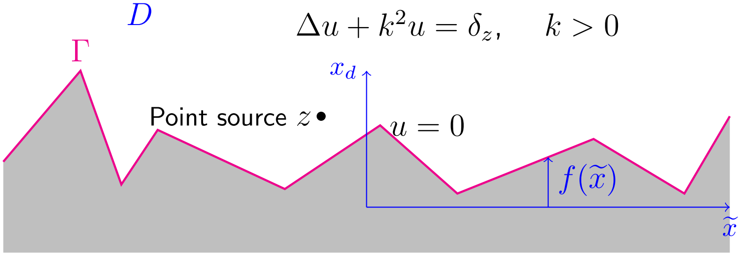

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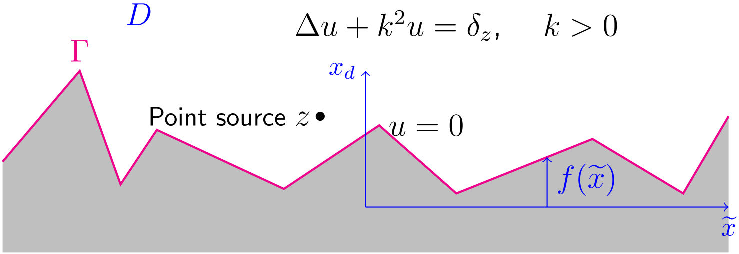

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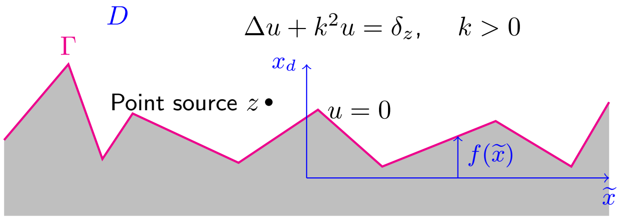
Applications in outdoor noise or radar propagation over ground and sea surfaces, and in optics: all nominally flat surfaces are rough at some scale!

Integral equation methods: rough surface scattering



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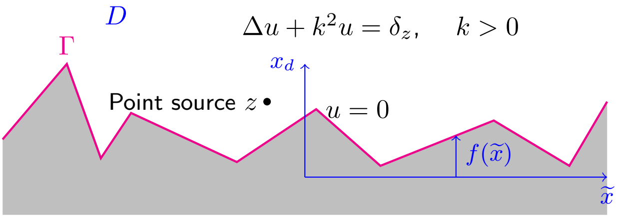

First idea: just use the **bounded obstacle formulation**, i.e.

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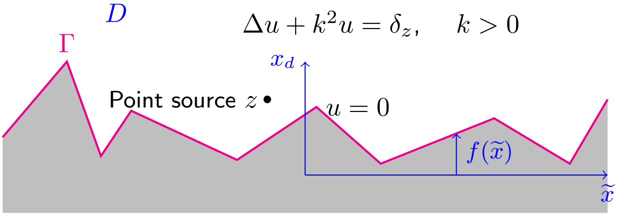

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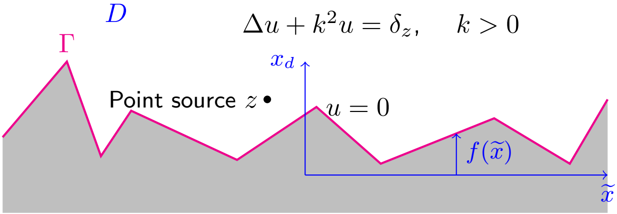

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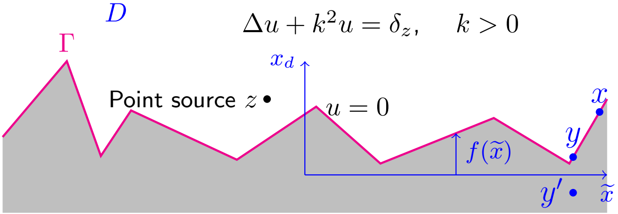

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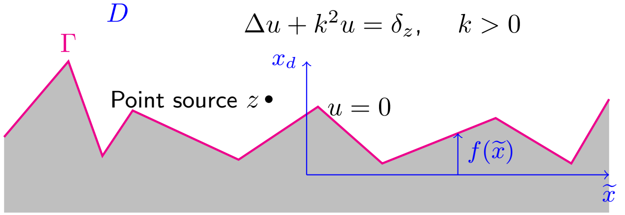
Solution: *(Zhang & C-W 2003, C-W, Heinemeyer, Potthast 2006a,b)* Replace $\Phi(x,y)$ with Dirichlet half-space Green's function, $G(x,y) := \Phi(x,y) - \Phi(x,y')$.



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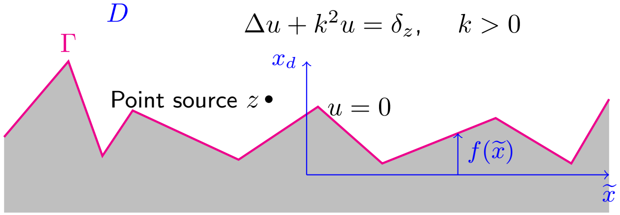

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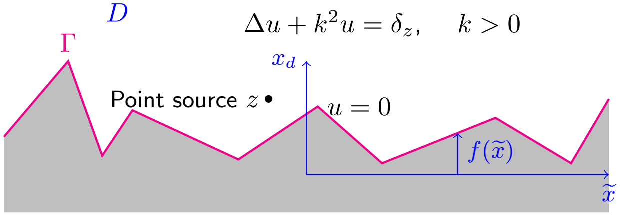
Theorem *(C-W, Heinemeyer, Potthast 2006a,b)*. A is bounded and invertible on $L^2(\Gamma)$, indeed, where L is the Lipschitz constant of f,

$$\|A^{-1}\| \leq 12(1+L)^2.$$



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Solution: *(Zhang & C-W 2003, C-W, Heinemeyer, Potthast 2006a,b)* Replace $\Phi(x,y)$ with Dirichlet half-space Green's function, $G(x,y) := \Phi(x,y) - \Phi(x,y')$.

Theorem *(C-W, Heinemeyer, Potthast 2006a,b)*. A is bounded and invertible on $L^2(\Gamma)$, indeed, where L is the Lipschitz constant of f,

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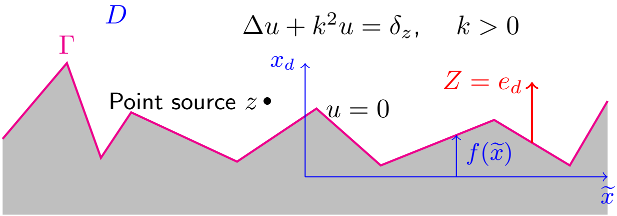

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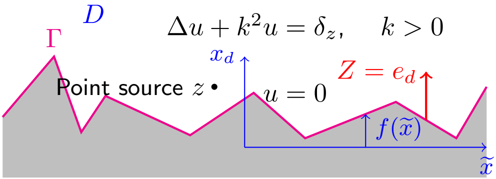

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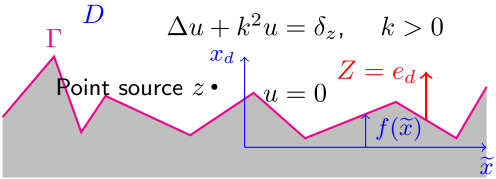

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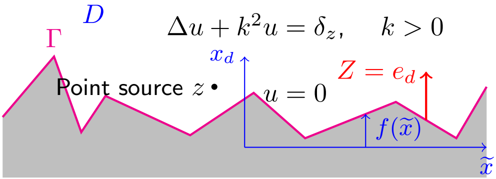

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- The convergence theory for Galerkin BEM and GMRES recalled earlier

Numerical results: flat Γ: $f(\tilde{x}) = f_- = 0.25$

2D numerical results when Γ is flat, applying h-BEM with P1 elements and uniform mesh on part of surface of length $2a$, with

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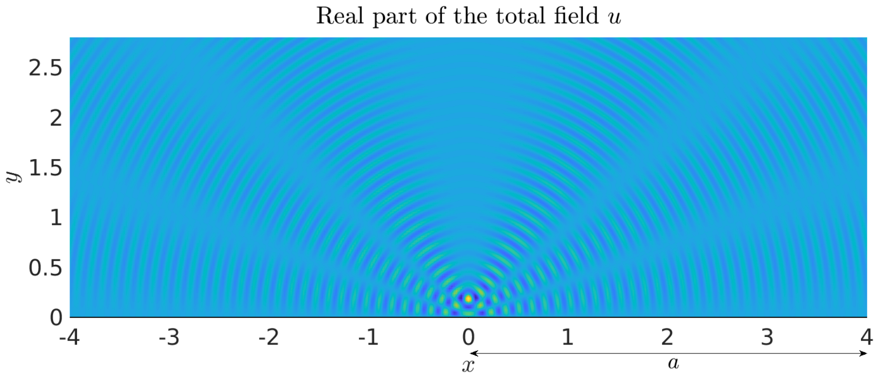

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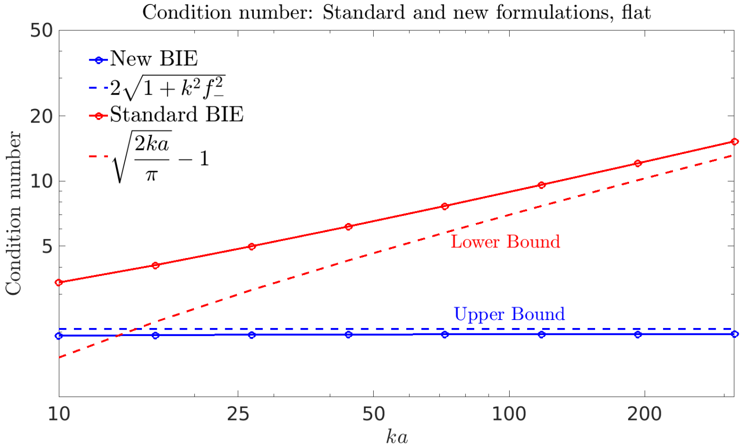

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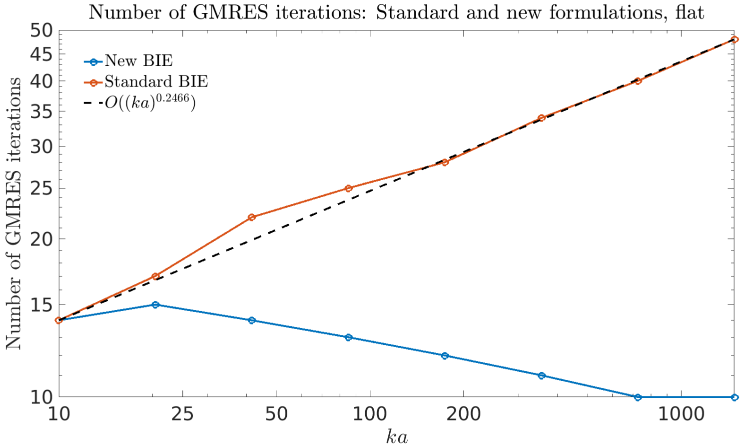

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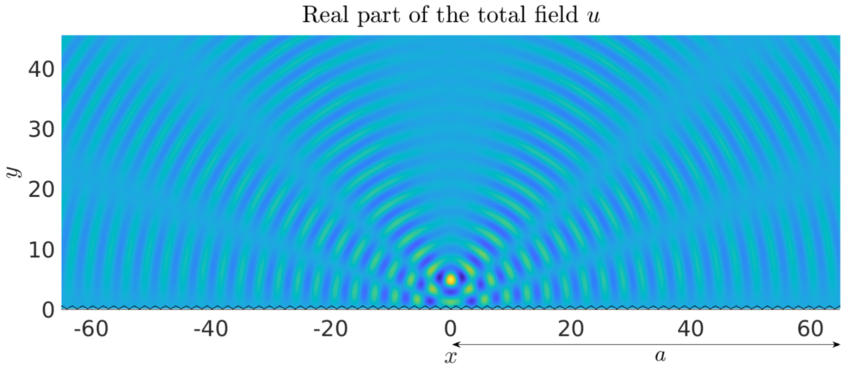

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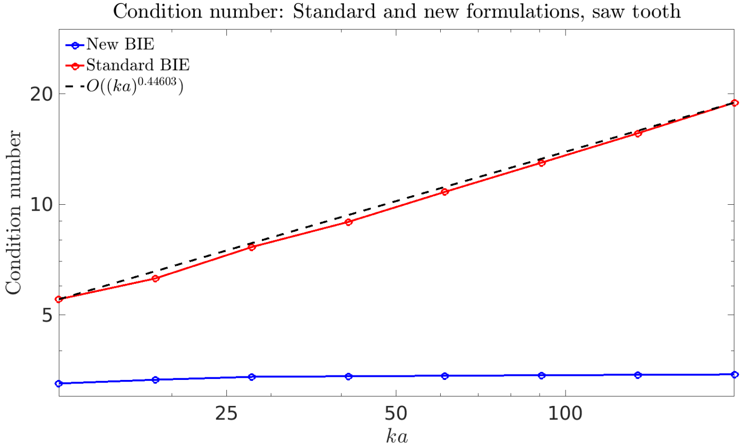

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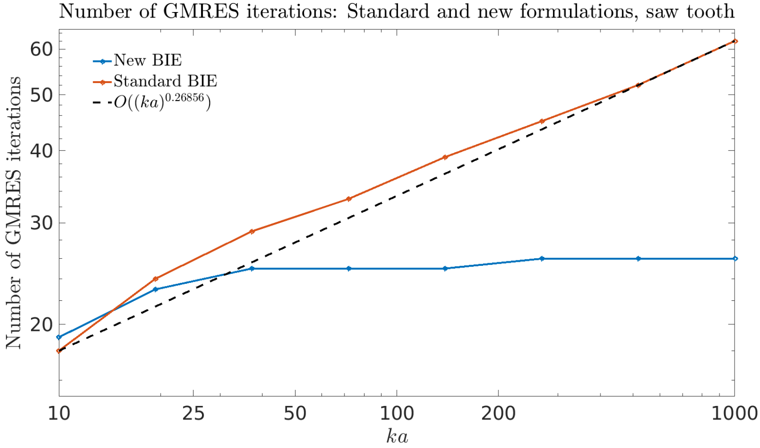

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- Proposed a new 2nd kind integral equation of this type for our RSS problem with $Z = e_d$, the constant vertical unit vector, for which A_Z is **bounded and uniformly-in-k coercive**, leading to proof of **convergence of combined surface truncation/Galerkin BEM**, and **convergence of GMRES** in a number of iterations independent of the element diameter h and the truncated surface diameter a.

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