High-frequency scattering by polygons and wedges via the complex-scaled (C-S) half-space matching method (HSMM)

Simon Chandler-Wilde

Department of Mathematics
and Statistics
University of Reading, UK

University of Reading

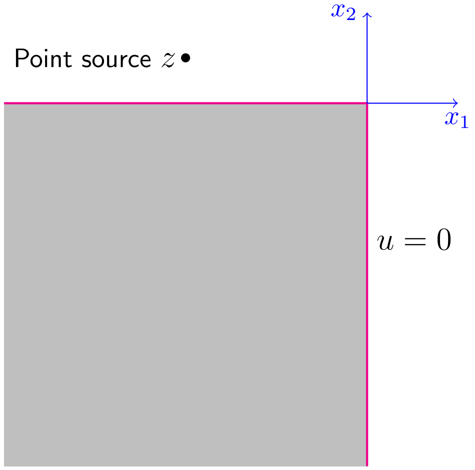
With: Anne-Sophie Bonnet-Bendhia & Sonia Fliss (ENSTA, France)

INI Canonical Scattering Workshop, February 2023

Diffraction by a (right-angled) wedge – the HSMM way

u satisfies S.R.C. at ∞

$$\Delta u + k^2 u = \delta_z, \quad k > 0$$



The Half-Space Matching Method Philosophy

1. It is easy to solve explicitly Dirichlet problems in half-planes.
2. So express your solution in each of a number of overlapping half-planes using this explicit solution.
3. The HSMM equations are obtained by **enforcing compatibility** between these different half-plane representations.

Bonnet-BenDhia, Fliss, Tonnoir, *J. Comp. Appl. Math.* 2018

The Half-Space Matching Method Philosophy

1. It is easy to solve explicitly Dirichlet problems in half-planes.
2. So express your solution in each of a number of overlapping half-planes using this explicit solution.
3. The HSMM equations are obtained by **enforcing compatibility** between these different half-plane representations.

Bonnet-BenDhia, Fliss, Tonnoir, *J. Comp. Appl. Math.* 2018

Step 1. Let's solve the simplest half-plane problem ...

u satisfies S.R.C. at ∞

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega$$

$$u = g \quad \text{on} \quad \Sigma$$

The Half-Space Matching Method Philosophy

1. It is easy to solve explicitly Dirichlet problems in half-planes.
2. So express your solution in each of a number of overlapping half-planes using this explicit solution.
3. The HSMM equations are obtained by **enforcing compatibility** between these different half-plane representations.

Bonnet-BenDhia, Fliss, Tonnoir, *J. Comp. Appl. Math.* 2018

Step 1. Let's solve the simplest half-plane problem ...

u satisfies S.R.C. at ∞

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega$$

$$u = g \quad \text{on} \quad \Sigma$$

Solution is

$$u(x) = 2\int_{\Sigma} \frac{\partial \Phi(x,y)}{\partial y_2} g(y)\, \mathrm{d}s(y), \quad x \in \Omega,$$

where

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|).$$

The Half-Space Matching Method Philosophy

1. It is easy to solve explicitly Dirichlet problems in half-planes.
2. So express your solution in each of a number of overlapping half-planes using this explicit solution.
3. The HSMM equations are obtained by **enforcing compatibility** between these different half-plane representations.

Bonnet-BenDhia, Fliss, Tonnoir, *J. Comp. Appl. Math.* 2018

Step 1. Let's solve the simplest half-plane problem ...

u satisfies S.R.C. at ∞

$$\Delta u + k^2 u = \delta_z \quad \text{in} \quad \Omega$$

Point source $z\bullet$ $\quad u = g \quad$ on $\quad \Sigma$

Solution is

$$u(x) = 2\int_{\Sigma} \frac{\partial \Phi(x,y)}{\partial y_2} g(y)\, \mathrm{d}s(y), \quad x \in \Omega,$$

where

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|).$$

The Half-Space Matching Method Philosophy

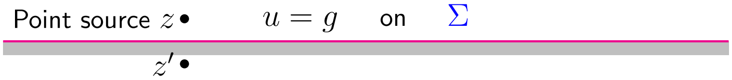
1. It is easy to solve explicitly Dirichlet problems in half-planes.
2. So express your solution in each of a number of overlapping half-planes using this explicit solution.
3. The HSMM equations are obtained by **enforcing compatibility** between these different half-plane representations.

Bonnet-BenDhia, Fliss, Tonnoir, *J. Comp. Appl. Math.* 2018

Step 1. Let's solve the simplest half-plane problem ...

u satisfies S.R.C. at ∞

$$\Delta u + k^2 u = \delta_z \quad \text{in} \quad \Omega$$



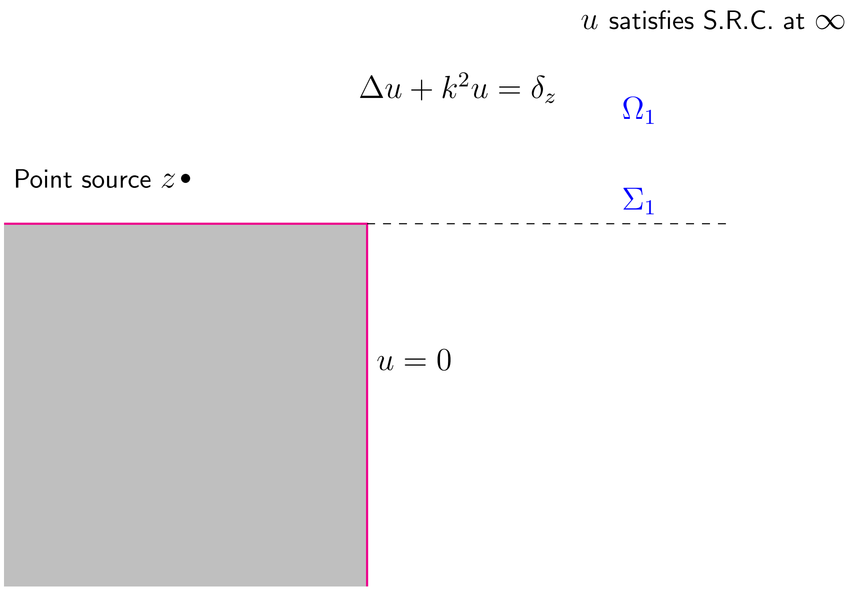
Solution is

$$u(x) = G(x,z) + 2\int_{\Sigma} \frac{\partial \Phi(x,y)}{\partial y_2} g(y)\,\mathrm{d}s(y), \quad x \in \Omega,$$

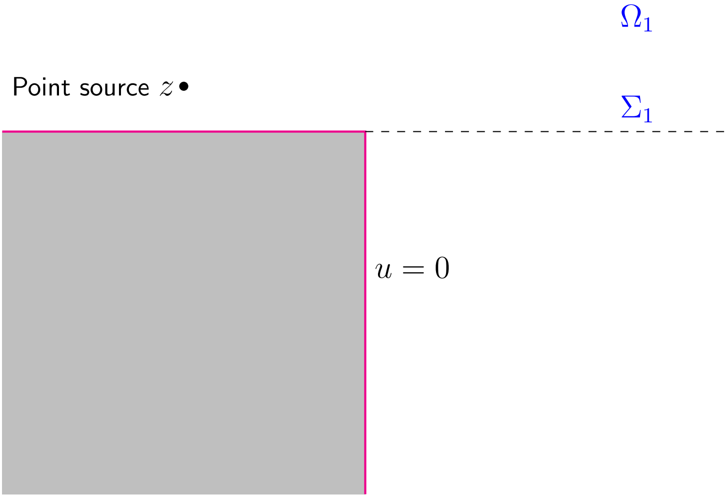
where

$$G(x,z) := \Phi(x,z) - \Phi(x,z'), \quad \Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|).$$

Diffraction by a (right-angled) wedge – the HSMM way

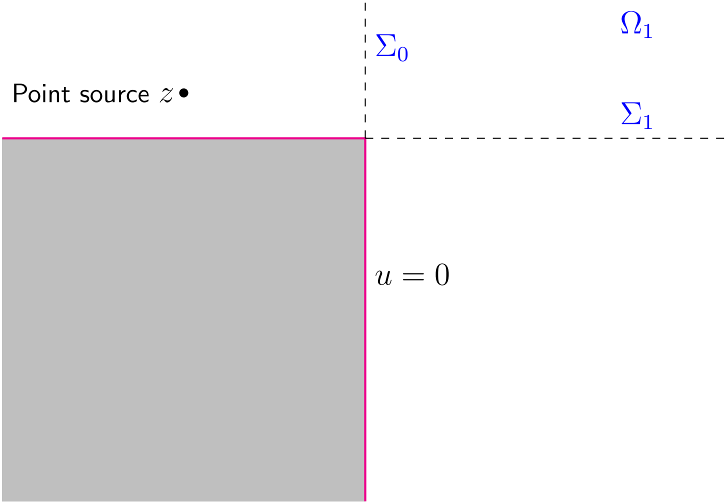


Diffraction by a (right-angled) wedge – the HSMM way



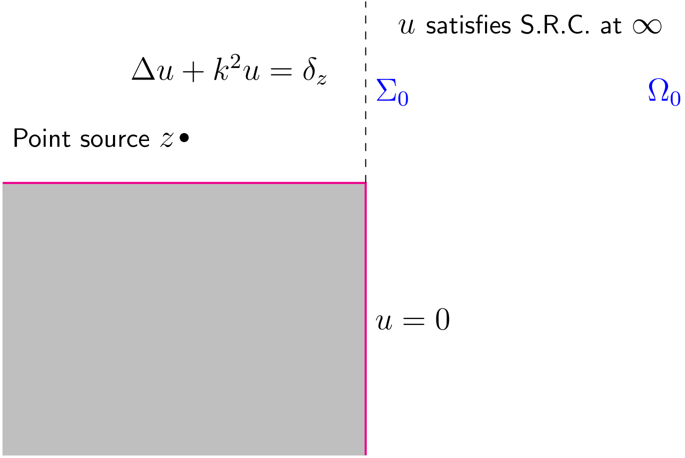
$$u(x) = G(x, z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x, y)}{\partial y_2} u(y)\,\mathrm{d}s(y), \quad x \in \Omega_1.$$

Diffraction by a (right-angled) wedge – the HSMM way

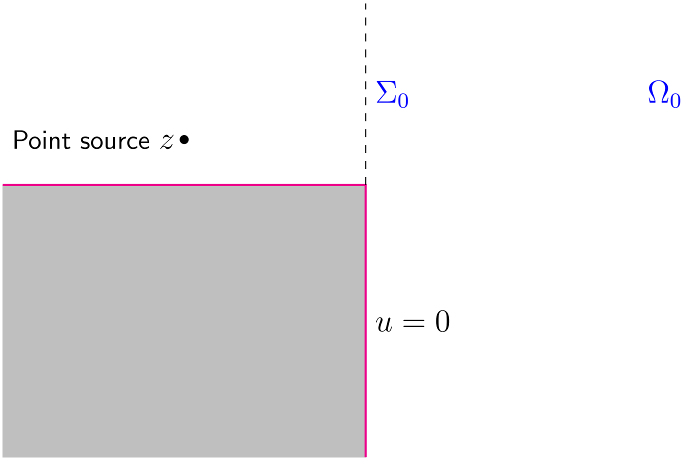


$$u(x) = G(x, z) + 2 \int_{\Sigma_1} \frac{\partial \Phi(x, y)}{\partial y_2} u(y)\, \mathrm{d}s(y), \quad x \in \Sigma_0.$$

Diffraction by a (right-angled) wedge – the HSMM way

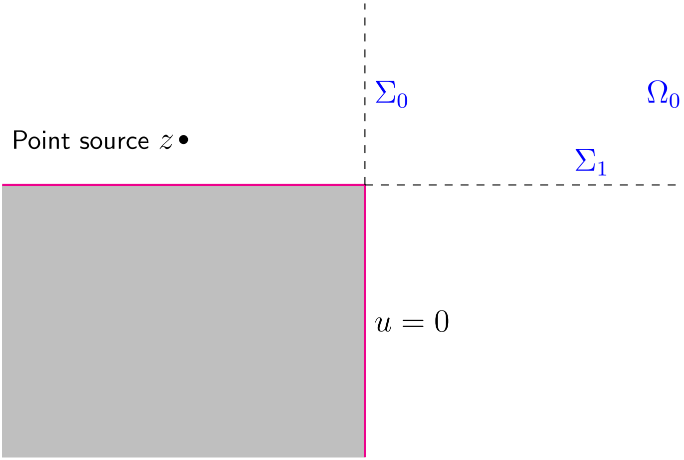


Diffraction by a (right-angled) wedge – the HSMM way



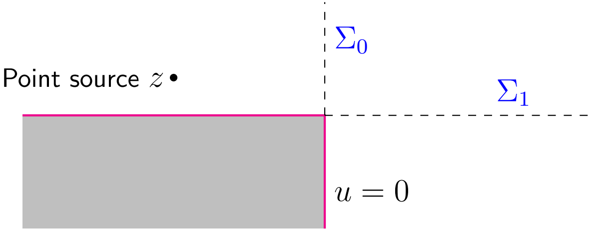
$$u(x) = 2\int_{\Sigma_0} \frac{\partial \Phi(x,y)}{\partial y_1} u(y)\,\mathrm{d}s(y), \quad x \in \Omega_0.$$

Diffraction by a (right-angled) wedge – the HSMM way



$$u(x) = 2 \int_{\Sigma_0} \frac{\partial \Phi(x,y)}{\partial y_1} u(y) \, \mathrm{d}s(y), \quad x \in \Sigma_1.$$

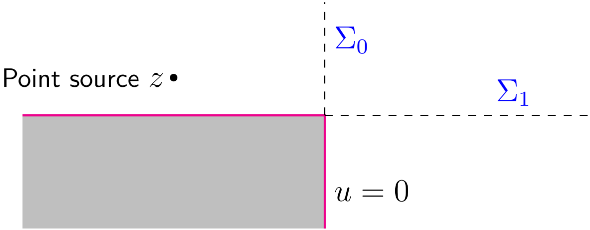
The HSMM integral equations



Two integral equations for unknowns $u|_{\Sigma_0}$ and $u|_{\Sigma_1}$:

$$
\begin{aligned}
u(x) &= G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_0, \\
u(x) &= 2\int_{\Sigma_0} \frac{\partial \Phi(x,y)}{\partial y_1} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_1.
\end{aligned}
$$

The HSMM integral equations



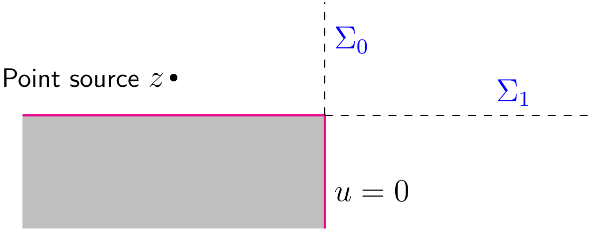
Two integral equations for unknowns $u|_{\Sigma_0}$ and $u|_{\Sigma_1}$:

$$u(x) = G(x,z) + 2\int_{\Sigma_1} \frac{\partial\Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_0,$$
$$u(x) = 2\int_{\Sigma_0} \frac{\partial\Phi(x,y)}{\partial y_1} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_1.$$

These equations have exactly one solution (Bonnet-BenDhia, C-W, Fliss, *SIAM J. Appl. Math.* 2022) if one requires, additionally, that

$$u(x) = a_m \mathrm{e}^{\mathrm{i}kr} r^{-1/2} + O(r^{-3/2}), \quad \text{as } r := |x| \to \infty \text{ with } x \in \Sigma_m, \quad m = 0, 1.$$

The HSMM integral equations



Two integral equations for unknowns $u|_{\Sigma_0}$ and $u|_{\Sigma_1}$:

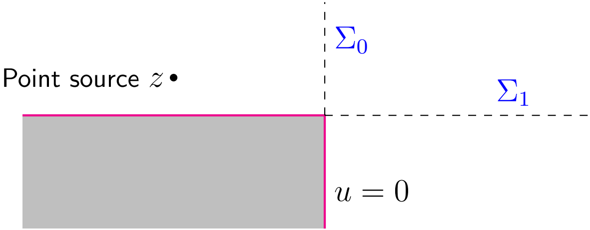
$$\begin{aligned} u(x) &= G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_0, \\ u(x) &= 2\int_{\Sigma_0} \frac{\partial \Phi(x,y)}{\partial y_1} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_1. \end{aligned}$$

These equations have exactly one solution (Bonnet-BenDhia, C-W, Fliss, *SIAM J. Appl. Math.* 2022) if one requires, additionally, that

$$u(x) = a_m \mathrm{e}^{\mathrm{i}kr} r^{-1/2} + O(r^{-3/2}), \quad \text{as } r := |x| \to \infty \text{ with } x \in \Sigma_m, \quad m = 0, 1.$$

Let $\varphi_0(s) := u((0,s))$ and $\varphi_1(s) := u((s,0))$, for $s \geq 0$.

The HSMM integral equations



Two integral equations for unknowns $u|_{\Sigma_0}$ and $u|_{\Sigma_1}$:

$$u(x) = G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_0,$$

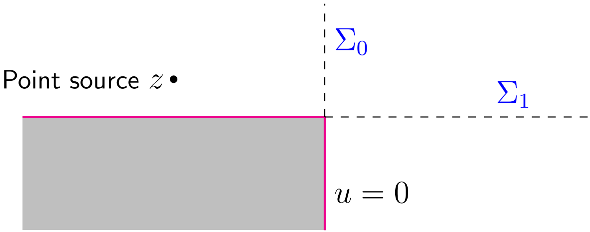
$$u(x) = 2\int_{\Sigma_0} \frac{\partial \Phi(x,y)}{\partial y_1} u(y)\,\mathrm{d}s(y), \quad x \in \Sigma_1.$$

These equations have exactly one solution (Bonnet-BenDhia, C-W, Fliss, *SIAM J. Appl. Math.* 2022) if one requires, additionally, that

$$u(x) = a_m \mathrm{e}^{\mathrm{i}kr} r^{-1/2} + O(r^{-3/2}), \quad \text{as } r := |x| \to \infty \text{ with } x \in \Sigma_m, \quad m = 0, 1.$$

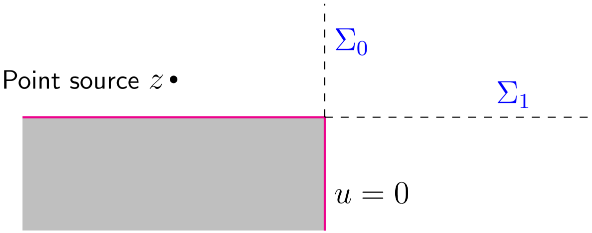
Let $\varphi_0(s) := u((0,s))$ and $\varphi_1(s) := u((s,0))$, for $s \geq 0$. Then, explicitly the above equations are ...

The HSMM integral equations



$$
\begin{aligned}
\varphi_0(s) &= \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s \geq 0,\\
\varphi_1(s) &= \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s \geq 0,
\end{aligned}
$$

The HSMM integral equations



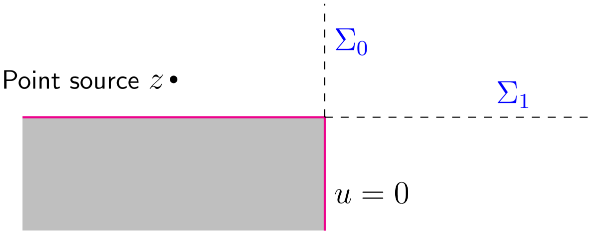
$$\varphi_0(s) = \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1(s) = \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s \geq 0,$$

with

$$\varphi_m(s) = a_m \mathrm{e}^{\mathrm{i}ks} s^{-1/2} + O(s^{-3/2}), \quad \text{as } s \to \infty, \quad m = 0, 1,$$

The HSMM integral equations



$$\varphi_0(s) = \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1(s) = \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s \geq 0,$$

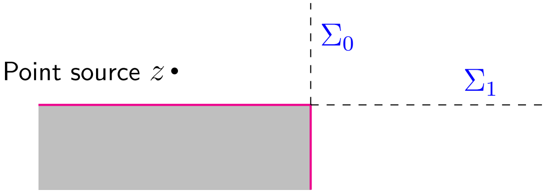
with

$$\varphi_m(s) = a_m \mathrm{e}^{\mathrm{i}ks} s^{-1/2} + O(s^{-3/2}), \quad \text{as } s \to \infty, \quad m = 0, 1,$$

and

$$\psi(s) := \frac{\mathrm{i}}{4}H_0^{(1)}\left(k\sqrt{(s-z_2)^2+z_1^2}\right) - \frac{\mathrm{i}}{4}H_0^{(1)}\left(k\sqrt{(s+z_2)^2+z_1^2}\right), \quad s \geq 0.$$

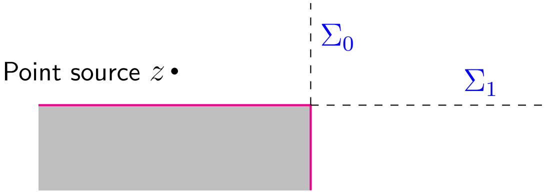
The **Complex-Scaled** HSMM integral equations



$$\begin{aligned} \varphi_0(s) &= \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s \geq 0, \\ \varphi_1(s) &= \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s \geq 0, \end{aligned}$$

$$\varphi_m(s) = a_m \mathrm{e}^{\mathrm{i}ks} s^{-1/2} + O(s^{-3/2}), \quad \text{as } s \to \infty, \quad m = 0, 1.$$

The **Complex-Scaled** HSMM integral equations

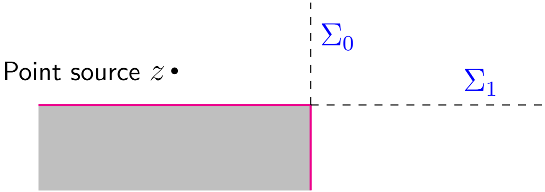


$$\begin{aligned}\varphi_0(s) &= \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s \geq 0,\\ \varphi_1(s) &= \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s \geq 0,\end{aligned}$$

$$\varphi_m(s) = a_m \mathrm{e}^{\mathrm{i}ks} s^{-1/2} + O(s^{-3/2}), \quad \text{as } s \to \infty, \quad m = 0, 1.$$

1. Each RHS provides an **analytic continuation** of the LHS into the right-hand complex plane

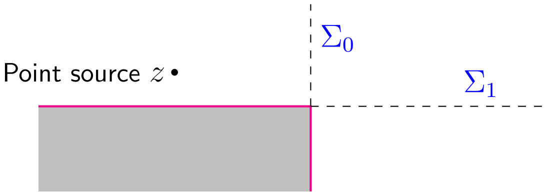
The **Complex-Scaled** HSMM integral equations



$$\begin{aligned}
\varphi_0(s) &= \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s = r\mathrm{e}^{\mathrm{i}\theta}, r \geq 0, \\
\varphi_1(s) &= \frac{\mathrm{i}ks}{2}\int_0^\infty \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s = r\mathrm{e}^{\mathrm{i}\theta}, r \geq 0,
\end{aligned}$$

1. Each RHS provides an **analytic continuation** of the LHS into the right-hand complex plane, so, for $0 < \theta < \pi/2$, ...

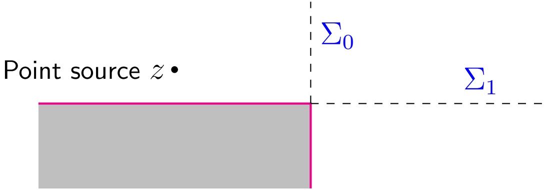
The **Complex-Scaled** HSMM integral equations



$$
\begin{aligned}
\varphi_0(s) &= \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s = r\mathrm{e}^{\mathrm{i}\theta}, r \geq 0,\\
\varphi_1(s) &= \frac{\mathrm{i}ks}{2}\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s = r\mathrm{e}^{\mathrm{i}\theta}, r \geq 0,
\end{aligned}
$$

1. Each RHS provides an **analytic continuation** of the LHS into the right-hand complex plane, so, for $0 < \theta < \pi/2$, ...
2. Rotating the paths of integration we get ...

The **Complex-Scaled** HSMM integral equations

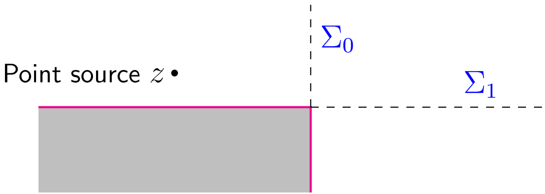


$$\varphi_0(s) = \psi(s) + \frac{\mathrm{i}ks}{2}\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1(t)\,\mathrm{d}t, \quad s = r\mathrm{e}^{\mathrm{i}\theta}, r \geq 0,$$

$$\varphi_1(s) = \frac{\mathrm{i}ks}{2}\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \frac{H_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0(t)\,\mathrm{d}t, \quad s = r\mathrm{e}^{\mathrm{i}\theta}, r \geq 0,$$

1. Each RHS provides an **analytic continuation** of the LHS into the right-hand complex plane, so, for $0 < \theta < \pi/2$, ...
2. Rotating the paths of integration we get ...
3. Introducing φ_m^θ and ψ^θ defined by $\varphi_m^\theta(r) := \varphi_m(r\mathrm{e}^{\mathrm{i}\theta})$ and $\psi^\theta(r) := \psi(r\mathrm{e}^{\mathrm{i}\theta})$, these equations are ...

The **Complex-Scaled** HSMM integral equations

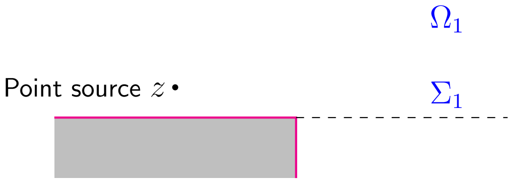


$$\varphi_0^{\theta}(s) = \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^{\infty}\frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1^{\theta}(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^{\infty}\frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.$$

1. Each RHS provides an **analytic continuation** of the LHS into the right-hand complex plane, so, for $0 < \theta < \pi/2$, ...
2. Rotating the paths of integration we get ...
3. Introducing φ_m^{θ} and ψ^{θ} defined by $\varphi_m^{\theta}(r) := \varphi_m(r\mathrm{e}^{\mathrm{i}\theta})$ and $\psi^{\theta}(r) := \psi(r\mathrm{e}^{\mathrm{i}\theta})$, these equations are ...

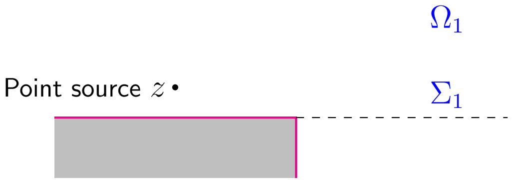
The **Complex-Scaled** HSMM integral equations



$$\varphi_0^{\theta}(s) = \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1^{\theta}(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.$$

The **Complex-Scaled** HSMM integral equations

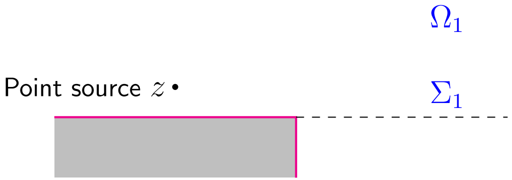


$$
\begin{aligned}
\varphi_0^{\theta}(s) &= \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,\\
\varphi_1^{\theta}(s) &= \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.
\end{aligned}
$$

We can recover u: for example for $x \in \Omega_1$,

$$
u(x) = G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y)
$$

The **Complex-Scaled** HSMM integral equations



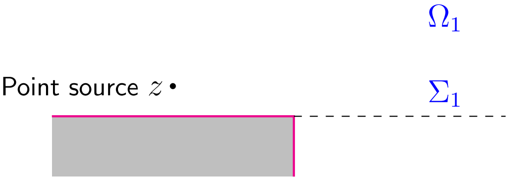
$$\varphi_0^{\theta}(s) = \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1^{\theta}(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.$$

We can recover u: for example for $x \in \Omega_1$,

$$\begin{aligned} u(x) &= G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y) \\ &= G(x,z) + \frac{\mathrm{i}kx_2}{2}\int_0^\infty \frac{H^{(1)}(k\sqrt{x_2^2+(t-x_1)^2})}{\sqrt{x_2^2+(t-x_1)^2}}\varphi_1(t)\,\mathrm{d}t \end{aligned}$$

The **Complex-Scaled** HSMM integral equations

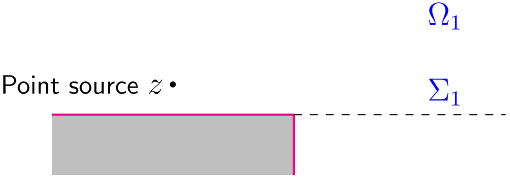


$$
\begin{aligned}
\varphi_0^{\theta}(s) &= \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,\\
\varphi_1^{\theta}(s) &= \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.
\end{aligned}
$$

We can recover u: for example for $x \in \Omega_1$,

$$
\begin{aligned}
u(x) &= G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y)\\
&= G(x,z) + \frac{\mathrm{i}kx_2}{2}\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \frac{H^{(1)}(k\sqrt{x_2^2+(t-x_1)^2})}{\sqrt{x_2^2+(t-x_1)^2}}\varphi_1(t)\,\mathrm{d}t
\end{aligned}
$$

The **Complex-Scaled** HSMM integral equations



$$\varphi_0^\theta(s) = \psi^\theta(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^\theta(t)\,\mathrm{d}t, \quad s \geq 0,$$

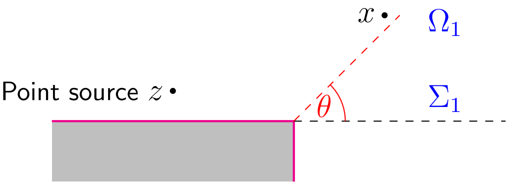
$$\varphi_1^\theta(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^\theta(t)\,\mathrm{d}t, \quad s \geq 0.$$

We can recover u: for example for $x \in \Omega_1$,

$$u(x) = G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y)$$

$$= G(x,z) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}x_2}{2}\int_0^\infty \frac{H^{(1)}(k\sqrt{x_2^2+(\mathrm{e}^{\mathrm{i}\theta}r-x_1)^2})}{\sqrt{x_2^2+(\mathrm{e}^{\mathrm{i}\theta}r-x_1)^2}}\varphi_1^\theta(r)\,\mathrm{d}r$$

The **Complex-Scaled** HSMM integral equations



$$\varphi_0^\theta(s) = \psi^\theta(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^\theta(t)\,\mathrm{d}t, \quad s \geq 0,$$

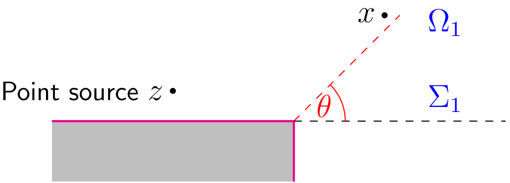
$$\varphi_1^\theta(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^\theta(t)\,\mathrm{d}t, \quad s \geq 0.$$

We can recover u: for example for $x \in \Omega_1$,

$$\begin{aligned} u(x) &= G(x,z) + 2\int_{\Sigma_1}\frac{\partial\Phi(x,y)}{\partial y_2}u(y)\,\mathrm{d}s(y) \\ &= G(x,z) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}x_2}{2}\int_0^\infty \frac{H^{(1)}(k\sqrt{x_2^2+(\mathrm{e}^{\mathrm{i}\theta}r-x_1)^2})}{\sqrt{x_2^2+(\mathrm{e}^{\mathrm{i}\theta}r-x_1)^2}}\varphi_1^\theta(r)\,\mathrm{d}r, \end{aligned}$$

as long as $x_2 > \tan(\theta)\,x_1$.

The **Complex-Scaled** HSMM integral equations



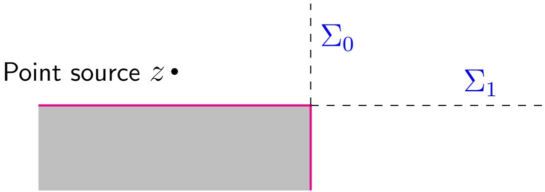
$$
\begin{aligned}
\varphi_0^{\theta}(s) &= \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta} s}{2} \int_0^{\infty} \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}} \varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0, \\
\varphi_1^{\theta}(s) &= \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta} s}{2} \int_0^{\infty} \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}} \varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.
\end{aligned}
$$

We can recover u: for example for $x \in \Omega_1$,

$$
\begin{aligned}
u(x) &= G(x,z) + 2\int_{\Sigma_1} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)\,\mathrm{d}s(y) \\
&= G(x,z) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta} x_2}{2} \int_0^{\infty} \frac{H^{(1)}(k\sqrt{x_2^2 + (\mathrm{e}^{\mathrm{i}\theta} r - x_1)^2})}{\sqrt{x_2^2 + (\mathrm{e}^{\mathrm{i}\theta} r - x_1)^2}} \varphi_1^{\theta}(r)\,\mathrm{d}r,
\end{aligned}
$$

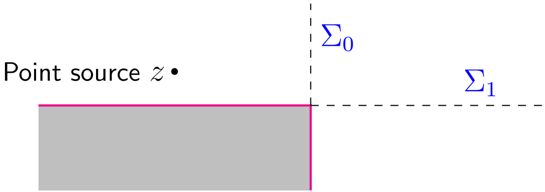
as long as $x_2 > \tan(\theta)\, x_1$. **So take** $\theta < \pi/4$.

But why use the **CS** HSMM integral equations?



$$\begin{aligned}
\varphi_0^{\theta}(s) &= \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,\\
\varphi_1^{\theta}(s) &= \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.
\end{aligned}$$

But why use the **CS** HSMM integral equations?



$$\varphi_0^\theta(s) = \psi^\theta(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^\theta(t)\,\mathrm{d}t, \quad s \ge 0,$$

$$\varphi_1^\theta(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^\theta(t)\,\mathrm{d}t, \quad s \ge 0.$$

Key feature. For some constant $C_\theta > 0$,

$$|\varphi_m^\theta(s)| \le C_\theta \exp(-k\sin(\theta)), \quad s \ge 0, \quad m = 0, 1.$$

But why use the **CS** HSMM integral equations?

$$\varphi_0^{\theta}(s) = \psi^{\theta}(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1^{\theta}(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^{\theta}(t)\,\mathrm{d}t, \quad s \geq 0.$$

But why use the **CS** HSMM integral equations?

$$\begin{aligned}\varphi_0^\theta(s) &= \psi^\theta(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^\theta(t)\,\mathrm{d}t, \quad s \geq 0,\\ \varphi_1^\theta(s) &= \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^\theta(t)\,\mathrm{d}t, \quad s \geq 0.\end{aligned}$$

Written in operator form these are

$$\varphi_0^\theta = \psi^\theta + D^\theta\varphi_1^\theta, \quad \varphi_1^\theta = D^\theta\varphi_0^\theta,$$

But why use the **CS** HSMM integral equations?

$$\varphi_0^\theta(s) = \psi^\theta(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^\theta(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1^\theta(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^\theta(t)\,\mathrm{d}t, \quad s \geq 0.$$

Written in operator form these are

$$\varphi_0^\theta = \psi^\theta + D^\theta\varphi_1^\theta, \quad \varphi_1^\theta = D^\theta\varphi_0^\theta, \quad \text{i.e.,} \quad \begin{pmatrix} \varphi_0^\theta \\ \varphi_1^\theta \end{pmatrix} = \begin{pmatrix} \psi^\theta \\ 0 \end{pmatrix} + \mathbf{D}^\theta \begin{pmatrix} \varphi_0^\theta \\ \varphi_1^\theta \end{pmatrix}$$

But why use the **CS** HSMM integral equations?

$$\varphi_0^\theta(s) = \psi^\theta(s) + \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_1^\theta(t)\,\mathrm{d}t, \quad s \geq 0,$$

$$\varphi_1^\theta(s) = \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}\theta}s}{2}\int_0^\infty \frac{H_1^{(1)}(k\mathrm{e}^{\mathrm{i}\theta}\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\varphi_0^\theta(t)\,\mathrm{d}t, \quad s \geq 0.$$

Written in operator form these are

$$\varphi_0^\theta = \psi^\theta + D^\theta\varphi_1^\theta, \quad \varphi_1^\theta = D^\theta\varphi_0^\theta, \quad \text{i.e.,} \quad \begin{pmatrix} \varphi_0^\theta \\ \varphi_1^\theta \end{pmatrix} = \begin{pmatrix} \psi^\theta \\ 0 \end{pmatrix} + \mathbf{D}^\theta \begin{pmatrix} \varphi_0^\theta \\ \varphi_1^\theta \end{pmatrix}$$

Theorem. As an operator on $L^2(\mathbb{R}_+)$, $D^\theta = D_0 + D_1^\theta$ where D_1^θ is compact and

$$\|D_0\| = \frac{1}{\sqrt{2}}, \quad \|D_1^\theta\| \leq \frac{\sqrt{1-\mathrm{e}^{-\pi\sin(\theta)}}}{4\sqrt{\pi}\sin(\theta)},$$

so that $\|\mathbf{D}^\theta\| = \|D^\theta\| \leq \|D_0\| + \|D_1^\theta\| < 1$ if

$$\theta > \sin^{-1}(p/\pi) \approx 0.13438\pi,$$

where p is the unique positive solution of

$$\pi - \pi\mathrm{e}^{-p} = 8(3-2\sqrt{2})p^2.$$

But why use the CS HSMM integral equations?

The equations in operator form are

$$\varphi_0^\theta = \psi^\theta + D^\theta \varphi_1^\theta, \quad \varphi_1^\theta = D^\theta \varphi_0^\theta$$

Theorem. As an operator on $L^2(\mathbb{R}_+)$, $D^\theta = D_0 + D_1^\theta$ where D_1^θ is compact and

$$\|D_0\| \le \frac{1}{\sqrt{2}}, \quad \|D_1^\theta\| \le \frac{\sqrt{1 - \mathrm{e}^{-\pi \sin(\theta)}}}{4\sqrt{\pi}\sin(\theta)},$$

so that $\|\mathbf{D}^\theta\| = \|D^\theta\| \le \|D_0\| + \|D_1^\theta\| < 1$ if

$$\theta > \sin^{-1}(p/\pi) \approx 0.13438\pi,$$

where p is the unique positive solution of

$$\pi - \pi\mathrm{e}^{-p} = 8(3 - 2\sqrt{2})p^2.$$

As a consequence, if

$$0.13438\pi < \theta < 0.25\pi,$$

u can be recovered from φ_0^θ and φ_1^θ, and $\|\mathbf{D}^\theta\| < 1$ so Neumann iteration converges

But why use the **CS** HSMM integral equations?

The equations in operator form are

$$\varphi_0^\theta = \psi^\theta + D^\theta \varphi_1^\theta, \quad \varphi_1^\theta = D^\theta \varphi_0^\theta$$

Theorem. As an operator on $L^2(\mathbb{R}_+)$, $D^\theta = D_0 + D_1^\theta$ where D_1^θ is compact and

$$\|D_0\| \leq \frac{1}{\sqrt{2}}, \quad \|D_1^\theta\| \leq \frac{\sqrt{1-\mathrm{e}^{-\pi\sin(\theta)}}}{4\sqrt{\pi}\sin(\theta)},$$

so that $\|\mathbf{D}^\theta\| = \|D^\theta\| \leq \|D_0\| + \|D_1^\theta\| < 1$ if

$$\theta > \sin^{-1}(p/\pi) \approx 0.13438\pi,$$

where p is the unique positive solution of

$$\pi - \pi\mathrm{e}^{-p} = 8(3-2\sqrt{2})p^2.$$

As a consequence, if

$$0.13438\pi < \theta < 0.25\pi,$$

u can be recovered from φ_0^θ and φ_1^θ, and $\|\mathbf{D}^\theta\| < 1$ so Neumann iteration converges, and Galerkin methods are convergent and quasi-optimal:

$$\text{Error in Galerkin solution } \leq \frac{\|\mathbf{D}^\theta\|}{1-\|\mathbf{D}^\theta\|} \text{ Best approximation from Galerkin subspace}$$

The **CS** HSMM integral equations: numerical results

The equations in operator form are

$$\varphi_0^{\theta} = \psi^{\theta} + D^{\theta}\varphi_1^{\theta}, \quad \varphi_1^{\theta} = D^{\theta}\varphi_0^{\theta}$$

The **CS** HSMM integral equations: numerical results

The equations in operator form are

$$\varphi_0^\theta = \psi^\theta + D^\theta \varphi_1^\theta, \quad \varphi_1^\theta = D^\theta \varphi_0^\theta$$

Approximate the integral operator D^θ by an $N \times N$ matrix D_N^θ by approximating

$$\int_0^\infty \approx \int_0^L \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

The **CS** HSMM integral equations: numerical results

The equations in operator form are

$$\varphi_0^\theta = \psi^\theta + D^\theta \varphi_1^\theta, \quad \varphi_1^\theta = D^\theta \varphi_0^\theta$$

Approximate the integral operator D^θ by an $N \times N$ matrix D_N^θ by approximating

$$\int_0^\infty \approx \int_0^L \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

The discrete unknowns are $N \times 1$ vectors $\boldsymbol{\varphi}_m^\theta$, $m = 0, 1$, approximations to the true values at the collocation points, that satisfy

$$\boldsymbol{\varphi}_0^\theta = \boldsymbol{\psi}^\theta + D_N^\theta \boldsymbol{\varphi}_1^\theta, \quad \boldsymbol{\varphi}_1^\theta = D_N^\theta \boldsymbol{\varphi}_0^\theta$$

The **CS** HSMM integral equations: numerical results

Approximate the integral operator D^{θ} by an $N \times N$ matrix D^{θ}_{N} by approximating

$$\int_0^{\infty} \approx \int_0^{L} \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

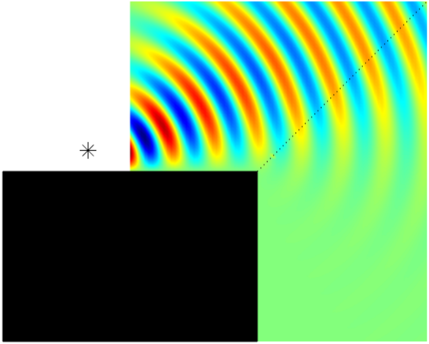
The CS HSMM integral equations: numerical results

Approximate the integral operator D^{θ} by an $N \times N$ matrix D_N^{θ} by approximating

$$\int_0^{\infty} \approx \int_0^{L} \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

Results for $L = 3$ wavelengths $= \frac{4\pi}{k}$, $\quad N = 20, \quad \theta = 0.24\pi$.



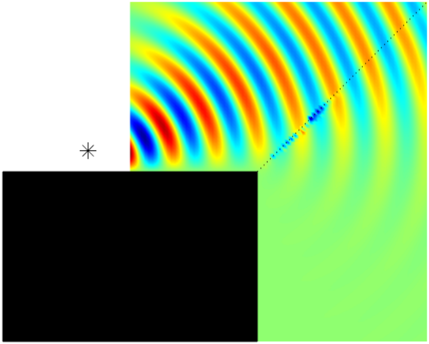
The **CS** HSMM integral equations: numerical results

Approximate the integral operator D^θ by an $N \times N$ matrix D_N^θ by approximating

$$\int_0^\infty \approx \int_0^L \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

Results for $L = 3$ wavelengths $= \frac{4\pi}{k}$, $\quad N = 20, \quad \theta = 0.27\pi$.



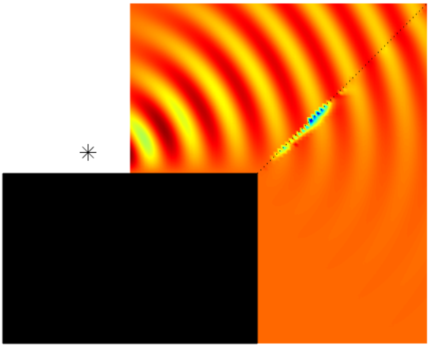
The CS HSMM integral equations: numerical results

Approximate the integral operator D^θ by an $N \times N$ matrix D_N^θ by approximating

$$\int_0^\infty \approx \int_0^L \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

Results for $L = 3$ wavelengths $= \dfrac{4\pi}{k}$, $\quad N = 20, \quad \theta = 0.29\pi.$



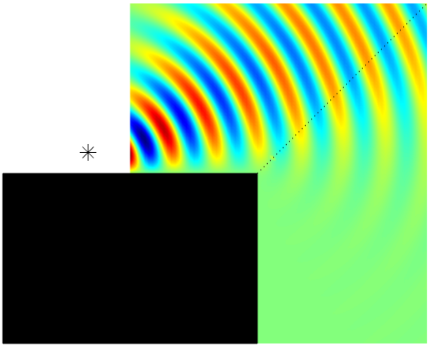
The CS HSMM integral equations: numerical results

Approximate the integral operator D^{θ} by an $N \times N$ matrix D_N^{θ} by approximating

$$\int_0^{\infty} \approx \int_0^{L} \approx \text{Midpoint rule with } N \text{ subintervals}$$

and by collocating at the midpoints of the subintervals.

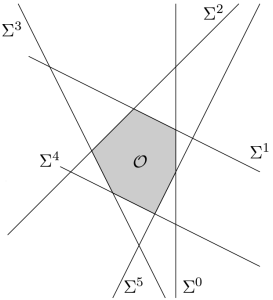
Results for $L = 3$ wavelengths $= \dfrac{4\pi}{k}$, $\quad N = 20, \quad \theta = 0.24\pi.$



What can the CS HSMM do apart from wedges?

What can the CS HSMM do apart from wedges?

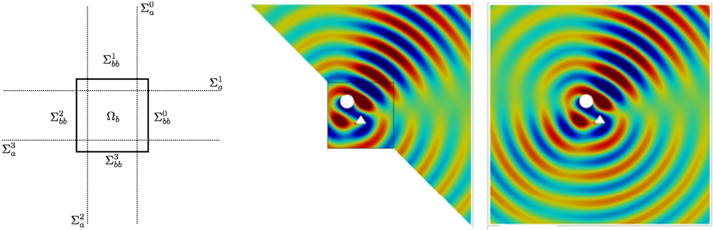
Polygons with Dirichlet (or other b.c.'s) in homogeneous medium



See Bonnet-Bendhia, C-W, Fliss et al, *SIAM J. Math. Anal.* 2022.

What can the CS HSMM do apart from wedges?

Arbitrary inhomogeneity in homogeneous medium



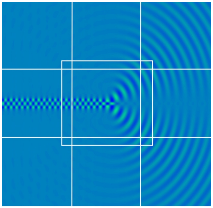
See Bonnet-Bendhia, C-W, Fliss et al, *SIAM J. Math. Anal.* 2022.

Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium

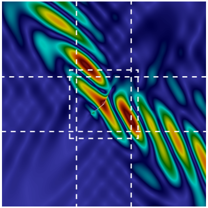
Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium
- The HSMM (without CS) already well-established for a range of scattering problems in complex media, e.g., **scalar problem with complex background**, Ott, Karlsruhe IT, PhD, 2017



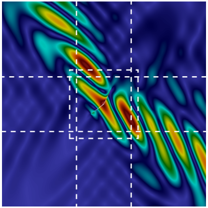
Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium
- The HSMM (without CS) already well-established for a range of scattering problems in complex media, e.g., **crack in anisotropic elastic medium**, Bécache, Bonnet-BenDhia, Fliss, Tonnoir, preprint, 2022



Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium
- The HSMM (without CS) already well-established for a range of scattering problems in complex media, e.g., **crack in anisotropic elastic medium**, Bécache, Bonnet-BenDhia, Fliss, Tonnoir, preprint, 2022

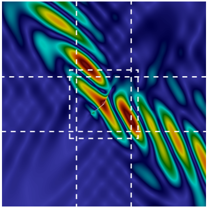


Open problems for the CS HSMM include:

- complete numerical analysis, and bounds for other wedge angles and b.c.'s;

Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium
- The HSMM (without CS) already well-established for a range of scattering problems in complex media, e.g., **crack in anisotropic elastic medium**, Bécache, Bonnet-BenDhia, Fliss, Tonnoir, preprint, 2022

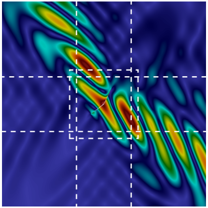


Open problems for the CS HSMM include:

- complete numerical analysis, and bounds for other wedge angles and b.c.'s;
- application of HSMM (and its CS version) to transmission wedge problems;

Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium
- The HSMM (without CS) already well-established for a range of scattering problems in complex media, e.g., **crack in anisotropic elastic medium**, Bécache, Bonnet-BenDhia, Fliss, Tonnoir, preprint, 2022

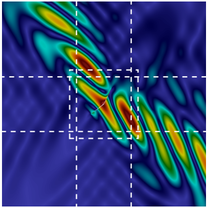


Open problems for the CS HSMM include:

- complete numerical analysis, and bounds for other wedge angles and b.c.'s;
- application of HSMM (and its CS version) to transmission wedge problems;
- exact solution of the CS HSMM integral equations by the K-L transform?

Conclusions and Open Problems

- The **CS HSMM** an attractive formulation for computation of scattering by wedges (with a variety of boundary conditions)
- The method equally attractive for scattering by polygons, indeed (through coupling to a local FEM solve) to any local perturbation of a homogeneous medium
- The HSMM (without CS) already well-established for a range of scattering problems in complex media, e.g., **crack in anisotropic elastic medium**, Bécache, Bonnet-BenDhia, Fliss, Tonnoir, preprint, 2022



Open problems for the CS HSMM include:

- complete numerical analysis, and bounds for other wedge angles and b.c.'s;
- application of HSMM (and its CS version) to transmission wedge problems;
- exact solution of the CS HSMM integral equations by the K-L transform?
- CS HSMM formulations for problems with more complex backgrounds.