IMA Journal of Numerical Analysis (2011) **31**, 1123−1153
doi:10.1093/imanum/drq009
Advance Access publication on August 20, 2010

A Nyström method for a boundary value problem arising in unsteady water wave problems

MARK D. PRESTON*
Department of Biology, University of York, PO Box 373, York YO10 5YW, UK
*Corresponding author: mdp505@york.ac.uk

AND

PETER G. CHAMBERLAIN AND SIMON N. CHANDLER-WILDE
Department of Mathematics, University of Reading, PO Box 220, Reading RG6 6AX, UK
p.g.chamberlain@reading.ac.uk s.n.chandler-wilde@reading.ac.uk

[Received on 3 September 2009; revised on 12 March 2010]

This paper is concerned with solving numerically the Dirichlet boundary value problem for Laplace's equation in a nonlocally perturbed half-plane. This problem arises in the simulation of classical unsteady water wave problems. The starting point for the numerical scheme is the boundary integral equation reformulation of this problem as an integral equation of the second kind on the real line in Preston *et al.* (2008, *J. Int. Equ. Appl.*, **20**, 121–152). We present a Nyström method for numerical solution of this integral equation and show stability and convergence, and we present and analyse a numerical scheme for computing the Dirichlet-to-Neumann map, i.e., for deducing the instantaneous fluid surface velocity from the velocity potential on the surface, a key computational step in unsteady water wave simulations. In particular, we show that our numerical schemes are superalgebraically convergent if the fluid surface is infinitely smooth. The theoretical results are illustrated by numerical experiments.

Keywords: water waves; Nyström method; Laplace's equation; nonperiodic surfaces.

1. The formulation of the water wave problem

The fluid motion in a classical two-dimensional water wave problem is well modelled as the motion under the influence of gravity of an incompressible, inviscid and irrotational fluid. As the fluid is irrotational, the flow can be described as a potential flow and the velocity **v** throughout the fluid is given by

$$\mathbf{v} = (v_1, v_2) = \nabla\phi, \tag{1.1}$$

where ϕ is the *velocity potential*. Under the standard assumptions of water wave theory, the velocity potential satisfies Laplace's equation in the fluid

$$\Delta\phi = 0, \tag{1.2}$$

and, in the absence of surface tension, Bernoulli's equation

$$\frac{\partial\phi}{\partial t} = -\frac{1}{2}|\nabla\phi|^2 - g x_2, \tag{1.3}$$

where x_2 is the vertical component of x, on the free surface.

We consider in this paper the case when, at each instant in time, the fluid occupies a perturbed half-plane domain of the form

$$\varOmega := \{(x_1, x_2)\colon x_2 < f(x_1), x_1 \in \mathbb{R}\},$$

where, for some constants f_- and f_+, the continuous function f satisfies

$$f_- \leqslant f(x_1) \leqslant f_+ \tag{1.4}$$

for $x_1 \in \mathbb{R}$, so that the fluid surface $\varGamma := \partial\varOmega = \{(x_1, f(x_1))\colon x_1 \in \mathbb{R}\}$ is the graph of a bounded function. We assume, moreover, some smoothness for $\varGamma$, namely that, for some $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the derivatives of f up to order $n+2$ exist and are bounded and continuous. At $x = (x_1, f(x_1)) \in \varGamma$ we define $\mathbf{n}(x) = (n_1(x), n_2(x))$ to be the unit normal vector directed out of $\varOmega$ and $\mathbf{s}(x) = (s_1(x), s_2(x))$ to be the unit tangent vector that has a positive horizontal component, $s_1(x) > 0$.

Given a set $G \subset \mathbb{R}^m$, where $m = 1$ or 2, let $BC(G)$ denote the set of real-valued functions on G that are bounded and continuous, a Banach space under the usual supremum norm. In terms of this notation, the main computational requirement in evolving the fluid boundary as a function of time is the solution to the following Dirichlet boundary value problem for ϕ. Given boundary data $\phi_\varGamma \in BC(\varGamma)$, find $\phi \in BC(\bar{\varOmega}) \cap C^2(\varOmega)$ such that

$$\varDelta\phi = 0 \;\text{ in } \varOmega \quad \text{and} \quad \phi = \phi_\varGamma \;\text{ on } \varGamma. \tag{1.5}$$

It was shown in Preston *et al.* (2008) that this boundary value problem is well-posed and that the solution satisfies the maximum principle

$$|\phi(x)| \leqslant \sup_{y \in \varGamma} |\phi_\varGamma(y)|, \quad x \in \varOmega.$$

A large part of this paper will be devoted to describing and analysing a numerical scheme for (1.5) that is a discretization of a boundary integral equation reformulation proposed recently in Preston *et al.* (2008). We will also discuss the numerical computation, by boundary integral equation methods, of the Dirichlet-to-Neumann map $\varLambda_\varGamma$, which is the map with input $\phi_\varGamma$ and output $\frac{\partial\phi}{\partial\mathbf{n}}$ on $\varGamma$, where ϕ is the solution to the above boundary value problem. Given this map, we can determine the velocity on the boundary by

$$\mathbf{v}|_\varGamma = \nabla\phi|_\varGamma = D\phi_\varGamma\mathbf{s} + \varLambda_\varGamma\phi_\varGamma\mathbf{n}, \tag{1.6}$$

where $D\phi_\varGamma = \frac{\partial\phi_\varGamma}{\partial s}$ is the tangential derivative of $\phi_\varGamma$. Hence we can evolve $\phi_\varGamma$ and the boundary $\varGamma$ (as the graph of a function f) using (1.3) and the kinematic boundary condition that the surface moves with the fluid. Precisely, on $\varGamma$, we have that f and $\mathbf{v} = (v_1, v_2)$ satisfy

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= -\frac{1}{2}|\mathbf{v}|^2 - gf, \\ \frac{\partial f}{\partial t} &= v_2 - v_1 f'. \end{aligned} \tag{1.7}$$

The formulation above separates the determination of the velocity potential (1.5) at any given time from the evolution of the two parameters, the boundary position and the Dirichlet boundary data (1.7). This separation naturally enables the system to be modelled by explicit time-stepping numerical methods used throughout the water wave literature, for example, Runge–Kutta and Adams–Bashforth schemes (see Beale *et al.*, 1996; Hou & Zhang, 2002; Baker & Beale, 2004; Fochesato & Dias, 2006; Xu & Guyenne, 2009). Subsequently, we will treat f and ϕ as only spatially dependent and suppress the time dependence, as we concentrate on solving the boundary value problem (1.5) and calculating the velocity (1.6) at a fixed time.

Let us spell out what the new contributions are in this paper. A main novelty is that this paper appears to be the first publication to tackle the numerical solution of the boundary value problem (1.5) in the general case of arbitrary bounded continuous Dirichlet data ϕ_Γ, with neither the boundary Γ nor ϕ_Γ assumed to be periodic. In the context of numerical simulation of periodic water waves, a numerical scheme, with a complete analysis, was provided in Hou & Zhang (2002) that applies to (1.5) in the special case when Γ and ϕ_Γ are periodic (so that, for some $S > 0$, we have $f(s+S) = f(s)$, where $s \in \mathbb{R}$). The boundary-integral-based scheme analysed in Hou & Zhang (2002) is one source of inspiration for the numerical method proposed and analysed in this paper. (The other is work on the numerical solution of acoustic rough surface scattering problems; Meier *et al.*, 2000; Meier, 2001; Meier & Chandler-Wilde, 2001; Haseloh, 2004.) But we note that the restriction to periodic Γ and boundary data in (1.5) simplifies the numerical scheme required and especially its analysis significantly. In particular, as we discuss later in the final section, with this periodicity, the operator in the boundary integral equation formulation we describe is a compact perturbation of the identity operator, so that stability and convergence of the type of scheme we propose follow, to a large extent, from standard arguments, for example, based on collectively compact operator theory (Atkinson, 1997).

A main motivation in designing an effective numerical scheme for (1.5) and for computing the Dirichlet-to-Neumann map is to provide a tool for the main computational problem at each time step for problems of simulation of nonperiodic water waves. We note, however, that our method does apply in the special case when the surface is periodic. An attraction of our numerical scheme and our analysis in that case is that it is clear from our results that our scheme is stable and convergent uniformly with respect to the period S. Thus the condition number of the linear system and the error in the numerical scheme remain bounded in the limit as $S \to \infty$. We also note that throughout we take care to prove stability results and error bounds that are uniform with respect to the surface Γ, provided that f lies in a certain constrained set, defined by the requirement (1.4) and by bounds on derivatives of f. Of course, our motivation here is again the application to the simulation of time-dependent water waves, where f varies in some constrained set as a function of time.

The structure of the paper is as follows. Section 2 recalls the integral equation formulation from Preston *et al.* (2008) that we will discretize. The main new results in this section are mapping properties of the integral operator, regularity results for the solution of the boundary integral equation and an explicit representation for and mapping properties of the Dirichlet-to-Neumann map. In Section 3 we turn to discretization and numerical analysis. Section 3.1 analyses a Nyström method for the boundary integral equation based on discretization of the integral operator, which is parameterized so that the integration is on the real line, by the trapezium rule. This analysis uses results from Meier (2001) and Meier & Chandler-Wilde (2001). In Section 3.2 we discuss a discrete approximation to the derivative of a continuously differentiable function on the real line based on localization and trigonometric interpolation. In Section 3.3 we use the methods and results of Section 3.2 to formulate and analyse an approximate Nyström method that is superalgebraically convergent when the Dirichlet data ϕ_Γ and Γ are smooth (in particular, $f \in C^\infty(\mathbb{R})$) but that does not require, as does the method of Section 3.1, access to the first and second derivatives of f but only access to sampled values of f on a uniform grid. Our intention is that this scheme in Section 3.3 should be of value in a time-stepping scheme for the water wave problem. In Section 3.4 we derive and analyse similar methods for approximating the

Dirichlet-to-Neumann map Λ_Γ, and hence for approximating the surface velocity $\mathbf{v}$. Finally, in Section 4 we illustrate the theoretical convergence results by numerical examples.

This paper is restricted to analysis of numerical methods connected to the two-dimensional water wave problem. There has been considerable recent interest in the numerical solution of (nonperiodic) three-dimensional water wave problems (e.g., Hou & Zhang, 2002; Fochesato & Dias, 2006; Xu & Guyenne, 2009), and it would be an interesting and nontrivial project to develop analogous numerical analysis to that presented in this paper for the three-dimensional case.

1.1 *Notation*

We collect here various notation used throughout, and, in particular, definitions of various function spaces that are necessary for the numerical analysis. Given an open or closed set $G \subset \mathbb{R}^m$, where $m = 1$ or 2, and $n \in \mathbb{N}_0$, let $BC^n(G)$ denote the set of functions ϕ: $G \to \mathbb{R}$ that are bounded and continuous and have (partial) derivatives up to order n that are all bounded and continuous. Here $BC^n(G)$ is a Banach space under the usual norm. We will abbreviate $BC^0(G)$ by $BC(G)$. For $0 < \alpha \leqslant 1$ let $BC^{0,\alpha}(G) \subset BC(G)$ denote the Banach space of functions that are bounded and uniformly Hölder continuous with index α and let $BC^{1,\alpha}(G)$ denote the Banach space of functions $\psi \in BC^1(G)$ for which $\nabla\psi \in BC^{0,\alpha}(G)$.

For $S > 0$ and $n \in \mathbb{N}_0$ let $BC^n_S(\mathbb{R}) \subset BC^n(\mathbb{R})$ denote the set of those functions $\phi \in BC^n(\mathbb{R})$ that are periodic with period S. We abbreviate $BC^0_S(\mathbb{R})$ by $BC_S(\mathbb{R})$ and let $BC^\infty_S(\mathbb{R}) := \bigcap_{n\in\mathbb{N}} BC^n_S(\mathbb{R})$. For $p > 0$ let $w_p(s) := (1+|s|)^p$, where $s \in \mathbb{R}$, and let $BC^n_p(\mathbb{R}) \subset BC^n(\mathbb{R})$ denote the Banach space

$$BC^n_p(\mathbb{R}) := \left\{ u \in BC^n(\mathbb{R}) \colon \|u\|_{BC^n_p(\mathbb{R})} := \sup_{m=0,\dots,n} \|w_p u^{(m)}\|_{BC^n(\mathbb{R})} < \infty \right\}$$

and let $BC^\infty_p(\mathbb{R}) := \bigcap_{n\in\mathbb{N}} BC^n_p(\mathbb{R})$.

Throughout, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ will be the standard unit coordinate vectors in $\mathbb{R}^3$. We will use the same notation $\mathbf{e}_1$ and $\mathbf{e}_2$ for the unit vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ in $\mathbb{R}^2$.

2. The boundary integral formulation and the Dirichlet-to-Neumann map

Choose $H > f_+$ and let Ω_H denote the half-plane $\Omega_H := \{(x_1,x_2)\colon x_1 \in \mathbb{R}, x_2 < H\}$ and let $\Gamma_H := \partial\Omega_H = \{(x_1,H)\colon x_1 \in \mathbb{R}\}$. Note that the half-plane Ω_H contains the perturbed half-plane domain Ω. We define the Dirichlet Green's function for the half-plane Ω_H by

$$\Phi_H(x,y) := \Phi(x,y) - \Phi(x,y^r), \quad x, y \in \mathbb{R}^2, \ \ x \neq y,$$

where

$$\Phi(x,y) := -\frac{1}{2\pi}\ln|x-y|$$

is the fundamental solution to Laplace's equation in two dimensions and $y^r := (y_1, 2H - y_2)$ is the reflection of y in Γ_H.

In Preston *et al.* (2008) it was proposed to look for a solution to the boundary value problem (1.5) in the form of a double-layer potential

$$\phi(x) := \int_\Gamma \frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)}\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y), \quad x \in \Omega, \tag{2.1}$$

for some density $\mu_\Gamma \in BC(\Gamma)$. Note that the half-plane Green's function is used in the definition (2.1) in place of the usual standard fundamental solution Φ. The following theorem was presented in Preston *et al.* (2008, Theorem 3.1).

THEOREM 2.1. The double-layer potential (2.1) with density $\mu_\Gamma \in BC(\Gamma)$ satisfies the boundary value problem (1.5) if and only if μ_Γ satisfies the second kind integral equation

$$\mu_\Gamma(x) - \int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \mu_\Gamma(y) \mathrm{d}\mathbf{s}(y) = -2\phi_\Gamma(x), \quad x \in \Gamma. \tag{2.2}$$

Defining the integral operator K_Γ by

$$(K_\Gamma \psi_\Gamma)(x) := 2\int_\Gamma \frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} \psi_\Gamma(y) \mathrm{d}\mathbf{s}(y),$$

we can rewrite (2.2) in operator notation as

$$(I - K_\Gamma)\mu_\Gamma = -2\phi_\Gamma.$$

The point of using Φ_H rather than Φ in (2.1) is that this choice ensures that the integrals (2.1) and (2.2) are well defined for all $\mu_\Gamma \in BC(\Gamma)$, and, indeed, that K_Γ is a bounded operator on $BC(\Gamma)$. From Preston *et al.* (2008, Theorem 3.4) we have, moreover, the following theorem on the boundedness of the inverse mapping $(I - K_\Gamma)^{-1}$.

THEOREM 2.2. The mapping $(I - K_\Gamma)$: $BC(\Gamma) \to BC(\Gamma)$ is invertible with a bounded inverse. Precisely, given $C_f > 0$, for some constant $C > 0$ depending only on $f_\pm$, H and C_f, it holds that

$$\|(I - K_\Gamma)^{-1}\| \leqslant C$$

whenever $\|f\|_{BC^2(\mathbb{R})} \leqslant C_f$.

It is convenient to introduce an isometric isomorphism J_Γ: $BC(\Gamma) \to BC(\mathbb{R})$, defined by $(J_\Gamma a_\Gamma)(\sigma) = a_\Gamma(\sigma, f(\sigma))$, where $\sigma \in \mathbb{R}$, for every $a_\Gamma \in BC(\Gamma)$. Let $\mu \in BC(\mathbb{R})$ be defined by $\mu := J_\Gamma \mu_\Gamma$, where μ_Γ is the solution of (2.2), let $\phi_0 \in BC(\mathbb{R})$ be defined by $\phi_0 := J_\Gamma \phi_\Gamma$ and let k_Ω be defined, for $x \in \mathbb{R}^2$ and $\sigma \in \mathbb{R}$, by

$$\begin{aligned} k_\Omega(x,\sigma) &= \left.\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\right|_{y=(\sigma, f(\sigma))} w(\sigma) \\ &= -\frac{1}{2\pi}\left(\frac{x-(\sigma, f(\sigma))}{|x-(\sigma, f(\sigma))|^2} - \frac{x-(\sigma, 2H-f(\sigma))}{|x-(\sigma, 2H-f(\sigma))|^2}\right)\cdot \mathbf{n}(\sigma) w(\sigma), \end{aligned} \tag{2.3}$$

where $w(\sigma) := \sqrt{1+f'(\sigma)^2}$ and $\mathbf{n}(\sigma) := \mathbf{n}((\sigma, f(\sigma))) = (-f'(\sigma), 1)/w(\sigma)$, and we note that $\mathbf{s}(\sigma) := \mathbf{s}((\sigma, f(\sigma))) = (1, f'(\sigma))/w(\sigma)$. We can then rewrite (2.1) as

$$\phi(x) = \int_\mathbb{R} k_\Omega(x,\sigma)\mu(\sigma)\mathrm{d}\sigma, \quad x \in \Omega, \tag{2.4}$$

and (2.2) as

$$\mu(\tau) - \int_\mathbb{R} k(\tau,\sigma)\mu(\sigma)\mathrm{d}\sigma = -2\phi_0(\tau), \quad \tau \in \mathbb{R}, \tag{2.5}$$

where $k(\tau,\sigma) := k_\Omega((\tau,\ f(\tau)),\sigma)$, for $\tau \neq \sigma$, while

$$k(\tau,\tau) = \frac{-1}{2\pi}\left(\frac{f''(\tau)}{\omega(\tau)^2} + \frac{1}{f(\tau)-H}\right), \quad \tau \in \mathbb{R}.$$

We will abbreviate (2.5) in operator form as

$$(I-K)\mu = -2\phi_0, \tag{2.6}$$

where $K := J_\Gamma K_\Gamma J_\Gamma^{-1}$ is the integral operator given by

$$(K\mu)(\tau) = \int_{\mathbb{R}} k(\tau,\sigma)\mu(\sigma)\mathrm{d}\sigma, \quad \tau \in \mathbb{R}.$$

We now prove a mapping property for the integral operator K and show that the smoothness of its kernel k is linked to the smoothness of the boundary. Let

$$r_1(\tau,\sigma) := \int_0^1 f'(\sigma + (\tau-\sigma)\xi)\mathrm{d}\xi$$

and

$$r_2(\tau,\sigma) := \int_0^1 f''(\sigma + (\tau-\sigma)\xi)(1-\xi)\mathrm{d}\xi \tag{2.7}$$

for $\tau,\sigma \in \mathbb{R}$, and note that, by Taylor's theorem (e.g., Hardy, 1958, pp. 327–328), for $f \in C^2(\mathbb{R})$ it holds that

$$f(\tau) = f(\sigma) + (\tau-\sigma)r_1(\tau,\sigma) = f(\sigma) + (\tau-\sigma)f'(\sigma) + (\tau-\sigma)^2 r_2(\tau,\sigma). \tag{2.8}$$

THEOREM 2.3. If $f \in BC^{n+2}(\mathbb{R})$ and $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ for some $n \in \mathbb{N}_0$ and $C_f > 0$, then $k \in BC^n(\mathbb{R}^2)$ and, for $i, j \in \mathbb{N}_0$ with $i + j \leqslant n$, we have

$$\left|\frac{\partial^{i+j}}{\partial\sigma^i\partial\tau^j}k(\tau,\sigma)\right| \leqslant \frac{C_k}{1+|\sigma-\tau|^2} \quad \text{for } \sigma,\tau \in \mathbb{R},$$

where C_k depends only on n, $f_\pm$, H and C_f. Furthermore, $K\colon BC(\mathbb{R}) \to BC^n(\mathbb{R})$ and there exists $C_K > 0$ depending only on n, $f_\pm$, H and C_f such that $\|K\| \leqslant C_K$.

Proof. For $\sigma,\tau \in \mathbb{R}^2$, where $\sigma \neq \tau$, by Taylor's theorem (Hardy, 1958) we have

$$\begin{aligned}\left.\frac{\partial\Phi(x,y)}{\partial\mathbf{n}(y)}\right|_{x=(\tau,f(\tau)),y=(\sigma,f(\sigma))} &= -\frac{1}{2\pi\,w(\sigma)}\frac{-(\tau-\sigma)f'(\sigma)+(f(\tau)-f(\sigma))}{(\tau-\sigma)^2+(f(\tau)-f(\sigma))^2}\\ &= -\frac{1}{2\pi\,w(\sigma)}\frac{r_2(\tau,\sigma)}{1+r_1(\tau,\sigma)^2}.\end{aligned} \tag{2.9}$$

Given $f \in BC^{n+2}(\mathbb{R})$, it is clear that $w \in BC^{n+1}(\mathbb{R})$, $r_1 \in BC^{n+1}(\mathbb{R}^2)$ and $r_2 \in BC^n(\mathbb{R}^2)$. Hence $k \in BC^n(\mathbb{R})$. Moreover, there exists a constant $C_k > 0$ that is dependent only on n, $f_\pm$, H and C_f such that $\|k\|_{BC^n(\mathbb{R})} \leqslant C_k$.

Now $\Phi_H(x, y)$ satisfies Laplace's equation as a function of both x and y in $\bar{\Omega}_H$ and, by Preston *et al.* (2008, Lemma 2.1), we have for $x, y \in \bar{\Omega}_H$ with $x \neq y$ and $y_2 \geqslant f_- - 1$ that

$$|\nabla_y \Phi_H(x, y)| \leqslant \frac{3(H - f_- + 1)}{\pi |x - y|^2}. \tag{2.10}$$

Then, from the regularity estimates in Gilbarg & Trudinger (1977, Theorem 3.9) for solutions to elliptic partial differential equations, where $\mathcal{D}_n \nabla_y \Phi_H(x, y)$ denotes any partial derivative of $\nabla_y \Phi_H(x, y)$ of order less than or equal to n with respect to the components of x and y, we have

$$|\mathcal{D}_n \nabla_y \Phi_H(x, y)| \leqslant \frac{C_n}{|x_1 - y_1|^2} \tag{2.11}$$

for $x, y \in \bar{\Omega}_H$, $|x_1 - y_1| \geqslant 1$ and $x_2, y_2 \in [f_-, f_+]$, where $C_n > 0$ depends only on n, $f_\pm$ and H. Since we have already shown that $\|k\|_{BC^n(\mathbb{R}^2)} \leqslant C_k$, it follows that, for some $C > 0$ depending only on n, $f_\pm$, H and C_f, we have

$$\left| \frac{\partial^{i+j}}{\partial \sigma^i \partial \tau^j} k(\tau, \sigma) \right| \leqslant \frac{C}{1 + |\sigma - \tau|^2}, \quad i + j \leqslant n,$$

for $\sigma, \tau \in \mathbb{R}$, as required.

The remainder of the result now follows from Meier *et al.* (2000, Theorem 2.4(a)). □

We now turn to the Dirichlet-to-Neumann map Λ_Γ. We first note that it was shown in Preston *et al.* (2008) that $I - K_\Gamma$ is also a bijection on $BC^{1,\alpha}(\Gamma)$ for $\alpha \in (0, 1)$ in the case that $f \in BC^2(\mathbb{R})$ and that, analogously to Theorem 2.2, as an operator on $BC^{1,\alpha}(\Gamma)$, we have

$$\|(I - K_\Gamma)^{-1}\| \leqslant C,$$

where C depends only on $f_\pm$, H and C_f. Furthermore, it was shown in Preston *et al.* (2008) that, if $\mu \in BC^{1,\alpha}(\Gamma)$, then ϕ given by (2.1) satisfies $\phi \in BC^{1,\alpha}(\bar{\Omega})$ with

$$\|\phi\|_{BC^{1,\alpha}(\bar{\Omega})} \leqslant C \|\mu\|_{BC^{1,\alpha}(\Gamma)},$$

where C, again, depends only on $f_\pm$, H and C_f. The above results, combined with Preston *et al.* (2008, Theorem 3.1), imply that the Dirichlet-to-Neumann map Λ_Γ is a bounded operator from $BC^{1,\alpha}(\Gamma)$ to $BC^{0,\alpha}(\Gamma)$ with $\|\Lambda_\Gamma\| \leqslant C_\Lambda$, where C_Λ depends only on $f_\pm$, H and C_f. Moreover, explicitly,

$$\left.\frac{\partial \phi}{\partial \mathbf{n}}\right|_\Gamma = \Lambda_\Gamma \phi_\Gamma = M_\Gamma (I - K_\Gamma)^{-1} \phi_\Gamma, \tag{2.12}$$

where the bounded operator $M_\Gamma\colon BC^{1,\alpha}(\Gamma) \to BC^{0,\alpha}(\Gamma)$ is given by

$$M_\Gamma \mu_\Gamma(x) = \frac{\partial}{\partial \mathbf{n}(x)} \int_\Gamma \frac{\partial \Phi_H(x, y)}{\partial \mathbf{n}(y)} \mu_\Gamma(y) \mathrm{d}s(y), \quad x \in \Gamma.$$

We now derive an alternative, more easily computable, expression for $M_\Gamma \mu_\Gamma$.

THEOREM 2.4. If $\mu_\Gamma \in BC^{1,\alpha}(\Gamma)$ then, for $x \in \Gamma$, we have

$$M_\Gamma \mu_\Gamma(x) = \int_\Gamma m_\Gamma[\mu_\Gamma](x,y)\mathrm{d}\mathbf{s}(y),$$

where

$$\begin{aligned} m_\Gamma[\mu_\Gamma](x,y) = {} & \frac{\partial \Phi_H(x,y)}{\partial \mathbf{s}(y)}\left(\mathbf{n}(x)\cdot\mathbf{n}(x)\frac{\partial \mu_\Gamma}{\partial s}(x) - \mathbf{n}(x)\cdot\mathbf{n}(y)\frac{\partial \mu_\Gamma}{\partial s}(y)\right) \\ & + \left(\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)}\mathbf{n}(x)\cdot\mathbf{s}(y) - \gamma(x,y)n_1(x)\right)\frac{\partial \mu_\Gamma}{\partial s}(y) \\ & + \left(\frac{\partial \gamma(x,y)}{\partial \mathbf{n}(x)}n_2(y) - \frac{\partial \gamma(x,y)}{\partial \mathbf{s}(x)}n_1(y)\right)\mu_\Gamma(y), \\ \gamma(x,y) = {} & \frac{x_2 + y_2 - 2H}{\pi|x - y^r|^2} \end{aligned}$$

and $\frac{\partial \mu_\Gamma}{\partial s}$ denotes the tangential derivative of μ_Γ.

Proof. Let $\mu_\Gamma \in BC^{1,\alpha}(\Gamma)$ and ϕ be the double-layer potential given by (2.1). Now, since $\frac{\partial}{\partial x_2}\Phi(x,y^r) = \frac{\partial}{\partial y_2}\Phi(x,y^r)$ and $\frac{\partial}{\partial x_1}\Phi(x,y^r) = -\frac{\partial}{\partial y_1}\Phi(x,y^r)$, it holds that

$$\nabla_x \Phi_H(x,y) + \nabla_y \Phi_H(x,y) = -2\mathbf{e}_2\frac{\partial}{\partial y_2}\Phi(x,y^r) = \gamma(x,y)\mathbf{e}_2 \tag{2.13}$$

and

$$\frac{\partial \Phi_H(x,y)}{\partial \mathbf{n}(y)} = -\nabla_x\cdot(\Phi_H(x,y)\mathbf{n}(y)) + n_2(y)\gamma(x,y).$$

It is convenient at this point to regard our two-dimensional vectors above as the first two components of three-dimensional vectors with zero third component in the direction of the standard coordinate direction $\mathbf{e}_3$ and note that $\mathbf{e}_3 = \mathbf{s}(y)\wedge\mathbf{n}(y) = \mathbf{e}_1\wedge\mathbf{e}_2$. This enables us to use the vector identity $\nabla\wedge\nabla\wedge A = -\Delta A + \nabla\nabla\cdot A$ and gives

$$\begin{aligned} \nabla\phi(x) = {} & -\int_\Gamma \nabla_x\wedge\nabla_x\wedge(\Phi_H(x,y)\mathbf{n}(y))\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y) \\ & + \int_\Gamma n_2(y)\nabla_x\gamma(x,y)\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y). \end{aligned}$$

Now, using (2.13), we have

$$\begin{aligned} \nabla_x\wedge(\Phi_H(x,y)\mathbf{n}(y)) &= -\mathbf{n}(y)\wedge\nabla_x\Phi_H(x,y) \\ &= \mathbf{n}(y)\wedge\nabla_y\Phi_H(x,y) - \gamma(x,y)\mathbf{n}(y)\wedge\mathbf{e}_2 \\ &= -\frac{\partial \Phi_H(x,y)}{\partial \mathbf{s}(y)}\mathbf{e}_3 - \gamma(x,y)n_1(y)\mathbf{e}_3 \end{aligned}$$

and

$$\nabla_x \wedge \nabla_x \wedge (\Phi_H(x,y)\mathbf{n}(y)) = \mathbf{e}_3 \wedge \frac{\partial}{\partial \mathbf{s}(y)} \nabla_x \Phi_H(x,y) + n_1(y)\mathbf{e}_3 \wedge \nabla_x \gamma(x,y).$$

Thus, interchanging the order of differentiation and then integrating by parts, we have

$$\nabla\phi(x) = \mathbf{e}_3 \wedge \int_\Gamma \nabla_x \Phi_H(x,y) \frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y)$$
$$- \int_\Gamma (n_1(y)\mathbf{e}_3 \wedge \nabla_x \gamma(x,y) - n_2(y)\nabla_x \gamma(x,y))\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y).$$

Clearly, the second integral is continuous in $\bar{\Omega}$ and, since $\frac{\partial \mu_\Gamma}{\partial \mathbf{s}} \in BC^{0,\alpha}(\Gamma)$, by applying Theorem 6.18 of Kress (1999) we see that the first integral can be continuously extended from Ω to $\bar{\Omega}$. Thus, taking the limit as x approaches Γ and using Theorem 6.18 of Kress (1999), we see that

$$\begin{aligned}
\frac{\partial\phi}{\partial\mathbf{n}}(x) &= \mathbf{n}(x) \cdot \left(\mathbf{e}_3 \wedge \int_\Gamma \nabla_x \Phi_H(x,y) \frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y)\right) \\
&\quad - \mathbf{n}(x) \cdot \left(\int_\Gamma (n_1(y)\mathbf{e}_3 \wedge \nabla_x \gamma(x,y) - n_2(y)\nabla_x \gamma(x,y))\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y)\right) \\
&= \mathbf{n}(x) \cdot \left(\mathbf{e}_3 \wedge \int_\Gamma (-\nabla_y \Phi_H(x,y) + \gamma(x,y)\mathbf{e}_2) \frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y)\right) \\
&\quad + \int_\Gamma \left(\frac{\partial\gamma(x,y)}{\partial\mathbf{n}(x)} n_2(y) - \frac{\partial\gamma(x,y)}{\partial\mathbf{s}(x)} n_1(y)\right)\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y),
\end{aligned}$$

where the first integrals in each line are to be understood as Cauchy principal values and note that we have applied (2.13) again. Now, splitting $\nabla_y \Phi_H(x,y)$ into its normal and tangential components, we have

$$\begin{aligned}
\frac{\partial\phi}{\partial\mathbf{n}}(x) &= -\mathbf{n}(x) \cdot \left(\mathbf{e}_3 \wedge \int_\Gamma \left(\mathbf{n}(y)\frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)} + \mathbf{s}(y)\frac{\partial\Phi_H(x,y)}{\partial\mathbf{s}(y)}\right) \frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y)\right) \\
&\quad - \int_\Gamma \gamma(x,y)n_1(y)\frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y) \\
&\quad + \int_\Gamma \left(\frac{\partial\gamma(x,y)}{\partial\mathbf{n}(x)} n_2(y) - \frac{\partial\gamma(x,y)}{\partial\mathbf{s}(x)} n_1(y)\right)\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y) \\
&= \int_\Gamma \left(\mathbf{n}(x)\cdot\mathbf{s}(y)\frac{\partial\Phi_H(x,y)}{\partial\mathbf{n}(y)} - \mathbf{n}(x)\cdot\mathbf{n}(y)\frac{\partial\Phi_H(x,y)}{\partial\mathbf{s}(y)}\right) \frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y) \\
&\quad - \int_\Gamma \gamma(x,y)n_1(y)\frac{\partial \mu_\Gamma}{\partial \mathbf{s}}(y)\mathrm{d}\mathbf{s}(y) \\
&\quad + \int_\Gamma \left(\frac{\partial\gamma(x,y)}{\partial\mathbf{n}(x)} n_2(y) - \frac{\partial\gamma(x,y)}{\partial\mathbf{s}(x)} n_1(y)\right)\mu_\Gamma(y)\mathrm{d}\mathbf{s}(y).
\end{aligned} \tag{2.14}$$

Finally, we have the identity

$$\int_{\Gamma} \frac{\partial \Phi_H(x,y)}{\partial \mathbf{s}(y)} \mathrm{d}\mathbf{s}(y) = 0, \quad x \in \Gamma,$$

where the integral is understood as a Cauchy principal value, and therefore we can subtract the term

$$\frac{\partial \mu_{\Gamma}}{\partial \mathbf{s}}(x) \int_{\Gamma} \frac{\partial \Phi_H(x,y)}{\partial \mathbf{s}(y)} \mathrm{d}\mathbf{s}(y)$$

from (2.14) and hence the result is proven. □

We now define the equivalent integral operator over $\mathbb{R}$ to M_{Γ}, namely $M\colon BC^{1,\alpha}(\mathbb{R}) \to BC^{0,\alpha}(\mathbb{R})$, given by $M := J_{\Gamma} M_{\Gamma} J_{\Gamma}^{-1}$. In the case that $f \in BC^2(\mathbb{R})$, for $\psi \in BC^2(\mathbb{R})$ and $\tau, \sigma \in \mathbb{R}$ let

$$\mathbf{p}_{\psi}(\sigma) := \frac{\mathbf{n}(\sigma)\psi'(\sigma)}{\omega(\sigma)}, \quad \mathbf{q}_{\psi}(\tau,\sigma) := \int_0^1 \mathbf{p}'_{\psi}(\sigma + (\tau-\sigma)\xi)\mathrm{d}\xi,$$

noting that

$$\mathbf{q}_{\psi}(\tau,\sigma) = \frac{\mathbf{p}_{\psi}(\tau) - \mathbf{p}_{\psi}(\sigma)}{\tau - \sigma}, \quad \sigma \neq \tau. \tag{2.15}$$

Furthermore, let

$$\begin{aligned} m[\psi](\tau,\sigma) &= m_{\Gamma}[J_{\Gamma}^{-1}\psi(\tau, f(\tau))](\sigma, f(\sigma))\omega(\sigma) \\ &= m_1[\psi](\tau,\sigma) + m_2[\psi](\tau,\sigma) + m_3[\psi](\tau,\sigma), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &m_1[\psi](\tau,\sigma) \\ &:= \begin{cases} \dfrac{1}{2\pi}\left(\dfrac{(\tau-\sigma, f(\tau)-f(\sigma))}{(\tau-\sigma)^2 + (f(\tau)-f(\sigma))^2} - \dfrac{(\tau-\sigma, 2H - f(\tau) - f(\sigma))}{(\tau-\sigma)^2 + (2H - f(\tau) - f(\sigma))^2}\right) & \\ \qquad \cdot (\mathbf{n}(\tau)\cdot(\mathbf{p}_{\psi}(\tau) - \mathbf{p}_{\psi}(\sigma))\mathbf{s}(\sigma) + \mathbf{n}(\tau)\cdot\mathbf{s}(\sigma)\mathbf{p}_{\psi}(\sigma)), & \sigma \neq \tau, \\ \dfrac{1}{2\pi\omega(\tau)}\mathbf{q}_{\psi}(\tau,\tau)\cdot\mathbf{n}(\tau) = \dfrac{1}{2\pi\omega(\tau)}\mathbf{p}'_{\psi}(\tau)\cdot\mathbf{n}(\tau), & \sigma = \tau, \end{cases} \\ &m_2[\psi](\tau,\sigma) \\ &:= \frac{1}{\pi}\left(\frac{(2(\tau-\sigma)(2H - f(\tau) - f(\sigma)), (\tau-\sigma)^2 + (2H - f(\tau) - f(\sigma))^2)}{((\tau-\sigma)^2 + (2H - f(\tau) - f(\sigma))^2)^2}\right) \\ &\qquad \cdot (\mathbf{n}(\tau)n_2(\sigma) + \mathbf{s}(\tau)n_1(\sigma))\omega(\sigma)\psi(\sigma) \end{aligned}$$

and

$$m_3[\psi](\tau,\sigma) := \frac{1}{\pi}\left(\frac{2H - f(\tau) - f(\sigma)}{((\tau-\sigma)^2 + (2H - f(\tau) - f(\sigma))^2)^2}\right) n_1(\tau)\psi'(\sigma).$$

Then, by Theorem 2.4, for $\tau \in \mathbb{R}$ we have

$$(M\mu)(\tau) = \int_{\mathbb{R}} m[\mu](\tau, \sigma)\mathrm{d}\sigma. \tag{2.17}$$

The Dirichlet-to-Neumann map, $\Lambda := J_\Gamma \Lambda_\Gamma J_\Gamma^{-1}$, is then given by

$$\Lambda = M(I - K)^{-1}.$$

We now prove a result similar to Theorem 2.3 by showing that the smoothness of $m[\mu](\cdot,\cdot)$ is dependent on the smoothness of f and μ and that the operator M maps $BC^{n+2}(\mathbb{R})$ continuously into $BC^n(\mathbb{R})$.

THEOREM 2.5. If $f \in BC^{n+2}(\mathbb{R})$, $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ and $\mu \in BC^{n+2}(\mathbb{R})$ for some $n \in \mathbb{N}_0$ and $C_f > 0$, then $m[\mu](\cdot,\cdot) \in BC^n(\mathbb{R}^2)$ and, for $i, j \in \mathbb{N}_0$ with $i + j \leqslant n$, we have

$$\left|\frac{\partial^{i+j}}{\partial\sigma^i \partial\tau^j} m[\mu](\tau,\sigma)\right| \leqslant \frac{C_m}{1 + |\sigma - \tau|^2}\|\mu\|_{BC^{n+2}(\mathbb{R})}, \quad \sigma, \tau \in \mathbb{R},$$

where C_m depends only on n, $f_\pm$, H and C_f. Furthermore, $M\colon BC^{n+2}(\mathbb{R}) \to BC^n(\mathbb{R})$ and there exists $C_M > 0$, depending only on n, $f_\pm$, H and C_f, such that $\|M\| \leqslant C_M$.

Proof. For $\tau, \sigma \in \mathbb{R}$ let $\mathbf{p}(\sigma) := \mathbf{p}_\mu(\sigma)$ and $\mathbf{q}(\tau,\sigma) := \mathbf{q}_\mu(\tau,\sigma)$ and write $m_1[\mu](\tau,\sigma)$ as

$$m_1[\mu](\tau,\sigma) = m_{1,1}[\mu](\tau,\sigma) + m_{1,2}[\mu](\tau,\sigma)$$

on recalling equation (2.16) and where

$$m_{1,1}[\mu](\tau,\sigma) := \begin{cases} \dfrac{1}{2\pi}\dfrac{(\tau-\sigma, f(\tau) - f(\sigma))}{(\tau-\sigma)^2 + (f(\tau) - f(\sigma))^2}\cdot \mathbf{s}(\sigma)(\mathbf{p}(\tau) - \mathbf{p}(\sigma))\cdot\mathbf{n}(\tau), & \\ & \sigma \neq \tau, \\ \dfrac{1}{2\pi\omega(\tau)}\mathbf{p}'(\tau)\cdot\mathbf{n}(\tau), & \sigma = \tau, \end{cases}$$

and

$$m_{1,2}[\mu](\tau,\sigma) := \frac{1}{2\pi}\frac{(\tau-\sigma, 2H - f(\tau) - f(\sigma))}{(\tau-\sigma)^2 + (2H - f(\tau) - f(\sigma))^2}\cdot\mathbf{s}(\sigma)(\mathbf{p}(\tau) - \mathbf{p}(\sigma))\cdot\mathbf{n}(\tau)$$

$$+ k(\tau,\sigma)\frac{\mathbf{n}(\tau)\cdot\mathbf{s}(\sigma)\psi'(\sigma)}{(\omega(\sigma))^2}.$$

If $f \in BC^{n+2}(\mathbb{R})$ then $n_1, n_2, s_1, s_2, w \in BC^{n+1}(\mathbb{R})$ and, by Theorem 2.3, $k \in BC^n(\mathbb{R}^2)$, which implies that $m_{1,2}[\mu](\cdot,\cdot), m_2[\mu](\cdot,\cdot), m_3[\mu](\cdot,\cdot) \in BC^n(\mathbb{R}^2)$.

It remains to show that $m_{1,1}[\mu](\cdot,\cdot)$ has the required continuity. Now, for $\tau, \sigma \in \mathbb{R}$, where $\tau \neq \sigma$, using (2.8) and (2.15), we have

$$m_{1,1}[\mu](\tau,\sigma) = \frac{1}{2\pi\omega(\sigma)}\frac{\tau - \sigma + (f(\tau) - f(\sigma))f'(\sigma)}{(\tau-\sigma)^2 + (f(\tau) - f(\sigma))^2}(\mathbf{p}(\tau) - \mathbf{p}(\sigma))\cdot\mathbf{n}(\tau)$$

$$= \frac{1}{2\pi\omega(\sigma)}\frac{1 + f'(\sigma)r_1(\tau,\sigma)}{1 + r_1(\tau,\sigma)^2}\mathbf{q}(\tau,\sigma)\cdot\mathbf{n}(\tau),$$

and, since $r_1(\tau,\tau) = f'(\tau)$ and $\mathbf{q}(\tau,\tau) = \mathbf{p}'(\tau)$, the same formula applies for $\tau = \sigma$. In the proof of Theorem 2.3 we have already observed that $r_1 \in BC^{n+1}(\mathbb{R}^2)$, and clearly $\mathbf{p} \in BC^{n+1}(\mathbb{R})$, so that $\mathbf{q} \in BC^n(\mathbb{R}^2)$. Thus $m_{1,1}[\mu](\cdot,\cdot) \in BC^n(\mathbb{R})$. So $m[\mu](\cdot,\cdot) \in BC^n(\mathbb{R}^2)$. Moreover, using the above proof of Theorem 2.3, we see that there exists a constant $C_m > 0$, depending only on n, $f_\pm$, H and C_f, such that $\|m[\mu](\cdot,\cdot)\|_{BC^n(\mathbb{R})} \leqslant C_m\|\mu\|_{BC^{n+2}(\mathbb{R})}$.

Since also the bounds (2.10) and (2.11) hold, we see that, for $\tau,\sigma \in \mathbb{R}$, we have

$$\left|\frac{\partial^{i+j}}{\partial\sigma^i\partial\tau^j}m[\mu](\tau,\sigma)\right| \leqslant \frac{C}{1+|\sigma-\tau|^2}\|\mu\|_{BC^{n+2}(\mathbb{R})}, \quad i+j \leqslant n,$$

where C depends only on n, $f_\pm$, H and C_f. Hence, by Meier *et al.* (2000, Theorem 2.4(a)) again (taking $b = m[\mu](\cdot,\cdot)$ and letting $\phi \equiv 1$ in the definition of M^b in the notation of Meier *et al.* (2000)), $M\colon BC^{n+2}(\mathbb{R}) \to BC^n(\mathbb{R})$ and $\|M\| \leqslant C_M$, as required. □

We can now rewrite the velocity on the surface, given by (1.6), with respect to the horizontal component of the surface by using the isometric isomorphism J_Γ. Let $\boldsymbol{\nu}\colon \mathbb{R} \to \mathbb{R}^2$ be defined by $\boldsymbol{\nu} = (\nu_1,\nu_2) := J_\Gamma \mathbf{v}|_\Gamma$. Then, from (1.6) and (2.17), we have

$$\boldsymbol{\nu}(\tau) = \frac{\phi_0'(\tau)}{\omega(\tau)}\mathbf{s}(\tau) + (M\mu)(\tau)\mathbf{n}(\tau), \quad \tau \in \mathbb{R}. \tag{2.18}$$

REMARK 2.6. It follows from Theorems 2.3 and 2.5 that, if the surface and boundary data are infinitely smooth (i.e., $f, \phi_0 \in BC^\infty(\mathbb{R})$), then the density and hence velocity, given by (2.18), are also smooth (i.e., $\mu \in BC^\infty(\mathbb{R})$ and $\boldsymbol{\nu} \in BC^\infty(\mathbb{R}) \times BC^\infty(\mathbb{R})$).

3. Discretization and the Nyström method

In this section we propose and analyse a discretization of the integral equation (2.6) and of the expression for the normal velocity (2.12). To carry out this discretization we need two operators, a numerical integration or quadrature operator to approximate the integrals found in (2.6) and (2.12) and a discrete derivative operator to determine approximations to f' and μ'. We initially consider a partially discrete system in which just the quadrature operator is applied and use results from Meier *et al.* (2000) to show stability and convergence for this initial scheme, where the key feature is the assumption that we can know or calculate k exactly. Then we define and analyse a more fully discrete scheme in which we use a trigonometric discrete derivative operator to numerically calculate $\tilde{k}$, an approximation to k. Throughout, the discretization step length will be $h := 2\pi/N$, for some even $N \in \mathbb{N}$.

3.1 *Quadrature operator and the initial Nyström scheme*

It is well known that the trapezium rule is superalgebraically convergent for C^∞ periodic functions. It is also superalgebraically convergent for C^∞ integrands defined on the whole real line (which decay sufficiently rapidly at infinity for the integral to be well defined). Let us define a trapezium rule quadrature operator by

$$I_h u := h\sum_{j\in\mathbb{Z}} u(jh).$$

The following theorem establishes superalgebraic convergence in the case $u \in BC_p^\infty(\mathbb{R})$.

LEMMA 3.1 (Meier *et al.*, 2000, Theorem 3.9). If $u \in BC_p^n(\mathbb{R})$, $n \in \mathbb{N}$, n is even and $p > 1$, then, for $h > 0$, we have

$$\left| \int_{-\infty}^{\infty} u(\sigma)\mathrm{d}\sigma - I_h u \right| \leqslant C \|u\|_{BC_p^n(\mathbb{R})} h^n,$$

where $C > 0$ depends only on n and p.

Applying I_h to (2.6), we define a Nyström method approximation $\mu_N \in BC^n(\mathbb{R})$ to μ by

$$\mu_N = \phi_0 + K_N \mu_N, \tag{3.1}$$

where

$$K_N \psi(\tau) := I_N(k(\tau, \cdot)\psi(\cdot)) = h \sum_{j \in \mathbb{Z}} k(\tau, jh)\psi(jh), \quad \tau \in \mathbb{R}.$$

Explicitly, (3.1) is

$$\mu_N(\tau) = \phi_0(\tau) + h \sum_{j \in \mathbb{Z}} k(\tau, jh)\mu_N(jh), \quad \tau \in \mathbb{R}. \tag{3.2}$$

The values $\mu_N(ih)$, where $i \in \mathbb{Z}$, are determined by setting $\tau = ih$ and solving the resultant infinite set of linear equations.

Meier *et al.* (2000) proved results on the convergence of Nyström methods for second kind integral equations of the form

$$x(\tau) = y(\tau) + \int_{-\infty}^{\infty} (a(\tau, \sigma) \ln|\tau - \sigma| + b(\tau, \sigma))x(\sigma)\mathrm{d}\sigma, \quad \tau \in \mathbb{R},$$

where $a, b \in C^n(\mathbb{R}^2)$ and $a(\tau, \sigma)$ and $b(\tau, \sigma)$ decay like $|\tau - \sigma|^{-p}$ as $|\tau - \sigma| \to \infty$ for some $p > 1$. We can apply the results of Meier *et al.* (2000) by taking $a = 0$ and $b = k \in BC^n(\mathbb{R}^2)$. Theorems 2.2 and 2.3 show that the two conditions C_n'' and E of Meier *et al.* (2000) are satisfied, and so the following three theorems on the stability and convergence of the Nyström approximation (3.1) follow from Theorems 2.2, 2.8 and 3.13 in Meier *et al.* (2000).

THEOREM 3.2. If $f \in BC^3(\mathbb{R})$ and $\|f\|_{BC^3(\mathbb{R})} \leqslant C_f$ for some $C_f \geqslant 0$, then $K_N \colon BC(\mathbb{R}) \to BC(\mathbb{R})$ is bounded and

$$\|K_N\| \leqslant C,$$

where C depends only on $f_\pm$, H and C_f.

THEOREM 3.3. If $f \in BC^3(\mathbb{R})$ and $\|f\|_{BC^3(\mathbb{R})} \leqslant C_f$ for some $C_f \geqslant 0$, then there exist $\bar{N} \in \mathbb{N}$ and $C > 0$ such that, for all $N \geqslant \bar{N}$, we have that $(I - K_N)^{-1} \colon BC(\mathbb{R}) \to BC(\mathbb{R})$ is bounded and

$$\|(I - K_N)^{-1}\| \leqslant C, \tag{3.3}$$

where C depends only on $f_\pm$, H and C_f. Furthermore, if $\phi_0 \in BC(\mathbb{R})$ then, for $N \geqslant \bar{N}$, (3.1) has a unique solution $\mu_N \in BC(\mathbb{R})$ and

$$\|\mu_N\|_{BC(\mathbb{R})} \leqslant C\|\phi_0\|_{BC(\mathbb{R})}.$$

THEOREM 3.4. If $f \in BC^{n+2}(\mathbb{R})$, $\phi_0 \in BC^n(\mathbb{R})$ and $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ for some $C_f \geqslant 0$ and $n \in \mathbb{N}_0$ with n even, then there exists $\bar{N} \in \mathbb{N}$ such that, for all $N \geqslant \bar{N}$, we have

$$\|\mu - \mu_N\|_{BC^n(\mathbb{R})} \leqslant C\|\phi_0\|_{BC^n(\mathbb{R})} h^n, \quad N \geqslant \bar{N},$$

for some $C > 0$ depending only on n, $f_\pm$, H and C_f.

3.2 *Discrete derivative operator*

It is convenient in this section to utilize the following summation notation:

$$\sum_{j=-N/2}^{N/2}{}'' u_j := \frac{1}{2}(u_{-N/2} + u_{N/2}) + \sum_{j=-N/2+1}^{N/2-1} u_j.$$

For $u \in BC_{2\pi}(\mathbb{R})$ let $u_h \in BC_{2\pi}(\mathbb{R})$ be the trigonometric polynomial given by

$$u_h(\sigma) = \sum_{k=-N/2}^{N/2}{}'' \hat{u}_k \,\mathrm{e}^{\mathrm{i}k\sigma}, \quad \sigma \in \mathbb{R}, \tag{3.4}$$

where the coefficients $\hat{u}_k$ are given by

$$\hat{u}_k = \frac{1}{N}\sum_{l=-N/2}^{N/2}{}'' u(lh)\mathrm{e}^{-\mathrm{i}lkh}, \quad k = -\frac{N}{2}, \ldots, \frac{N}{2}.$$

It is a standard result that u_h interpolates u at jh, where $j \in \mathbb{Z}$, i.e., $u_h(jh) = u(jh)$, where $j \in \mathbb{Z}$. We can use the fast Fourier transform to calculate the coefficients $\hat{u}_k$.

THEOREM 3.5 (Meinardus, 1967, Theorem 41). If $u \in BC^n_{2\pi}(\mathbb{R})$ and u_h is defined by (3.4), then

$$\|u - u_h\|_{BC^m_{2\pi}(\mathbb{R})} \leqslant C_n\|u\|_{BC^n(\mathbb{R})} h^{n-m}$$

for $m = 0, 1, \ldots, n-1$, where the constant $C_n > 0$ depends only on n. In particular, if $u \in BC^\infty_{2\pi}(\mathbb{R})$ then u_h exhibits superalgebraic convergence, i.e., $\|u - u_h\|_{BC_{2\pi}(\mathbb{R})} = o(h^n)$ as $h \to \infty$ for all $n \in \mathbb{N}$.

Let us define a discrete approximate mth-order differential operator $\dot{D}^m_h\colon BC_{2\pi}(\mathbb{R}) \to BC_{2\pi}(\mathbb{R})$, for $m \in \mathbb{N}_0$, by

$$\dot{D}^m_h u(\sigma) := u^{(m)}_h(\sigma) = \sum_{k=-N/2}^{N/2}{}'' (\mathrm{i}k)^m \hat{u}_k \,\mathrm{e}^{\mathrm{i}k\sigma}, \quad \sigma \in \mathbb{R}. \tag{3.5}$$

Note that $\dot{D}^0_h u = u_h$. We now investigate the accuracy of $\dot{D}^m_h u$ as an approximation to the mth derivative of u. The following follows immediately from Theorem 3.5.

COROLLARY 3.6. If $u \in BC^n_{2\pi}(\mathbb{R})$ then, for $m = 1, \ldots, n-1$, we have

$$\|u^{(m)} - \dot{D}^m_h u\|_{BC_{2\pi}(\mathbb{R})} \leqslant C_n \|u\|_{BC^n_{2\pi}(\mathbb{R})} h^{n-m}.$$

Let $\chi \in BC^\infty(\mathbb{R})$ be a 'cut-off' function, compactly supported about 0, satisfying $0 \leqslant \chi(\sigma) \leqslant 1$, $\chi(\sigma) = \chi(-\sigma)$, $\chi(\sigma) = 0$ if $|\sigma| \geqslant \pi$ and $\chi(\sigma) = 1$ if $|\sigma| \leqslant 1$, where $\sigma \in \mathbb{R}$. We further define the translation operator $T_\sigma\colon BC(\mathbb{R}) \to BC(\mathbb{R})$ by $(T_\sigma u)(\tau) = u(\tau - \sigma)$ for $\sigma, \tau \in \mathbb{R}$, and a 2π-periodic extension operator $E\colon BC(\mathbb{R}) \to L^\infty(\mathbb{R})$ by the requirements that $(Eu)(\sigma) = u(\sigma)$, where $-\pi < \sigma \leqslant \pi$, and $(Eu)(\sigma + 2\pi) = (Eu)(\sigma)$, where $\sigma \in \mathbb{R}$. Using E, χ, T_σ and $\dot{D}^m_h$, we can define a discrete differential operator D^m_h on $BC(\mathbb{R})$ by

$$(D^m_h u)(\sigma) = (\dot{D}^m_h E(\chi T_\sigma u))(0), \quad \sigma \in \mathbb{R}. \tag{3.6}$$

THEOREM 3.7. If $u \in BC^n(\mathbb{R})$ then, for $m = 1, \ldots, n-1$, we have

$$\|u^{(m)} - D^m_h u\|_{BC(\mathbb{R})} \leqslant C_n \|u\|_{BC^n(\mathbb{R})} h^{n-m},$$

where C_n depends only on n and χ.

Proof. The operator $T_\sigma\colon BC^n(\mathbb{R}) \to BC^n(\mathbb{R})$, where $\sigma \in \mathbb{R}$, is bounded with $\|T_\sigma u\|_{BC^n(\mathbb{R})} = \|u\|_{BC^n(\mathbb{R})}$ for $\sigma \in \mathbb{R}$. The mapping $BC^n(\mathbb{R}) \to BC^n_{2\pi}(\mathbb{R})$, $u \to E(\chi u)$ is bounded with $\|E(\chi u)\|_{BC^n_{2\pi}(\mathbb{R})} \leqslant C\|u\|_{BC^n(\mathbb{R})}$, where C depends only on n and χ. Hence the mapping $BC^n(\mathbb{R}) \to BC^n_{2\pi}(\mathbb{R})$, $u \to E(\chi T_\sigma u)$ is bounded with $\|E(\chi T_\sigma u)\|_{BC^n_{2\pi}(\mathbb{R})} \leqslant C\|u\|_{BC^n(\mathbb{R})}$, where C depends only on n and χ. Furthermore, for all $\sigma \in \mathbb{R}$ we have $u(\sigma + \delta) = E(\chi T_\sigma u)(\delta)$, where $|\delta| \leqslant 1$. Therefore, by Corollary 3.6, the results hold. □

From the definition of the discrete derivative operator, through equations (3.5) and (3.6), it is clear that, for $u \in BC(\mathbb{R})$ and $m \in \mathbb{N}_0$, the values $D^m_h u(jh)$, where $j \in \mathbb{Z}$, depend only on the values of $u(x)$ at $x = jh$, where $j \in \mathbb{Z}$. To make this explicit, for $\tilde{u} = \{\tilde{u}_j\}_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ we define $E_N\tilde{u} \in BC(\mathbb{R})$ to be the piecewise linear function satisfying $E_N\tilde{u}(jh) = \tilde{u}_j$, where $j \in \mathbb{Z}$. Define $\tilde{D}^m_h\colon l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ by

$$(\tilde{D}^m_h \tilde{u})_j = D^m_h E_N \tilde{u}(jh), \quad j \in \mathbb{Z}. \tag{3.7}$$

Then, explicitly,

$$(\tilde{D}^m_h \tilde{u})_j = \sum_{k=-N/2}^{N/2}{}'' (\mathrm{i}k)^m c_k, \tag{3.8}$$

where

$$c_k = \frac{1}{N} \sum_{l=-N/2}^{N/2}{}'' \chi(lh) \mathrm{e}^{-\mathrm{i}klh} \tilde{u}_{l-j}.$$

In Section 3.4 we will need to approximate derivatives from *approximations* to functions at the interpolation points jh. The final theorem of this section details how this additional approximation affects the accuracy of the discrete derivative operator.

THEOREM 3.8. Suppose that $u \in BC^n(\mathbb{R})$ for some $n \in \mathbb{N}$, that, for $N \in \mathbb{N}$, we have $\tilde{u}_N := \{\tilde{u}_{j,N}\}_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ with $\tilde{u}_{j,N} \approx u(jh)$ and that, for some $p \in \mathbb{N}$ and $C_1 > 0$, we have

$$\max_{j\in\mathbb{Z}} |u(jh) - \tilde{u}_{j,N}| \leqslant C_1 \|u\|_{BC^n(\mathbb{R})} h^p$$

(where $h = 2\pi/N$). Then, for $m = 1, \ldots, n-1$, we have

$$\max_{j\in\mathbb{Z}} |u^{(m)}(jh) - (\tilde{D}_h^m \tilde{u}_N)_j| \leqslant C \|u\|_{BC^n(\mathbb{R})} h^q,$$

where $q = \min\{n - m, p - m - 1\}$ and C depends only on n, C_1 and χ.

Proof. By (3.7) and (3.8) and as $\|\chi\|_{BC(\mathbb{R})} = 1$, we have

$$\begin{aligned}
&\max_{j\in\mathbb{Z}} |D_h^m u(jh) - (\tilde{D}_h^m \tilde{u_N})_j| \\
&\quad = \max_{j\in\mathbb{Z}} \left| \frac{1}{N} \sum_{k=-N/2}^{N/2}{}'' \sum_{l=-N/2}^{N/2}{}'' (\mathrm{i}k)^m \chi(lh)(u((l-j)h) - \tilde{u}_{l-j,N}) \mathrm{e}^{-\mathrm{i}hkl} \right| \\
&\quad \leqslant C_1 \|u\|_{BC^n(\mathbb{R})} h^p \frac{1}{N} \sum_{k=-N/2}^{N/2}{}'' \sum_{l=-N/2}^{N/2}{}'' |k|^m \\
&\quad \leqslant 2^{-m} C_1 \|u\|_{BC^n(\mathbb{R})} h^p N^{m+1} \leqslant 2\pi^{n+1} C_1 \|u\|_{BC^n(\mathbb{R})} h^{p-m-1}.
\end{aligned}$$

Combining this inequality with Theorem 3.7, it follows that

$$\begin{aligned}
&\max_{j\in\mathbb{Z}} |u^{(m)}(jh) - (\tilde{D}_h^m \tilde{u}_N)_j| \\
&\quad \leqslant \|u^{(m)} - D_h^m u\|_{BC(\mathbb{R})} + \max_{j\in\mathbb{Z}} |D_h^m u(jh) - (\tilde{D}_h^m \tilde{u}_N)_j| \\
&\quad \leqslant C_n \|u\|_{BC^n(\mathbb{R})} h^{n-m} + 2\pi^{n+1} C_1 \|u\|_{BC^n(\mathbb{R})} h^{p-m-1},
\end{aligned}$$

where C_n is defined as in Theorem 3.7. □

3.3 *The fully discrete Nyström scheme*

We now define a numerical approximation to the kernel k of the integral equation by applying the differential operator (3.6) to approximate the derivatives f' and f'' by $D_h f$ and $D_h^2 f$, respectively. Thus our approximation is defined for $\tau, \sigma \in \mathbb{R}$ by

$$\tilde{k}(\tau,\sigma) = \begin{cases} \dfrac{-1}{\pi} \dfrac{(\tau-\sigma)D_h f(\sigma) - (f(\sigma) - f(\tau))}{(\tau-\sigma)^2 + (f(\tau) - f(\sigma))^2} + \tilde{k}^r(\tau,\sigma), & \sigma \neq \tau, \\ \dfrac{-1}{2\pi} \dfrac{D_h^2 f(\tau)}{1 + (D_h f(\tau))^2} + \tilde{k}^r(\tau,\tau), & \sigma = \tau, \end{cases}$$

where

$$\tilde{k}^r(\tau,\sigma) = \frac{1}{\pi}\frac{(\tau-\sigma)D_h f(\sigma) - (2H - f(\sigma) - f(\tau))}{(\tau-\sigma)^2 + (2H - f(\tau) - f(\sigma))^2}.$$

The fact that the function k is bounded relies on $((\tau, f(\tau)) - (\sigma, f(\sigma)))$ and $\mathbf{n}(\sigma)$ being perpendicular to each other in the limit as $\tau \to \sigma$. The vector $((\tau, f(\tau)) - (\sigma, f(\sigma)))$ is not necessarily perpendicular in the limit $\tau \to \sigma$ to the approximation to $\mathbf{n}(\sigma)$ obtained by replacing f' by $D_h f$. Hence $\tilde{k}$ is not necessarily bounded and the convergence analysis of Theorems 3.3 and 3.4 does not hold when replacing k by $\tilde{k}$. For this reason we now work on a discrete level.

For $N \in \mathbb{N}$ let $L_N\colon BC(\mathbb{R}) \to l^\infty(\mathbb{Z})$ be the restriction mapping defined by $L_N\psi = \{\psi(jh)\colon j \in \mathbb{Z}\}$ for $\psi \in BC(\mathbb{R})$. Clearly, $\|L_N\psi\|_\infty \leqslant \|\psi\|_{BC(\mathbb{R})}$. Recalling that $\phi_0 = J_\Gamma\phi_\Gamma$ is the inhomogeneous term in (2.5), let $\phi_N = L_N\phi_0 = \{\phi_0(jh)\}_{j\in\mathbb{Z}} = \{\phi_j\}_{j\in\mathbb{Z}}$.

For $j \in \mathbb{Z}$ let $\mathbf{x}_j = (jh, f(jh))$ and $\mathbf{x}_j^r = (jh, 2H - f(jh))$ and let

$$\omega_j = \sqrt{1 + (D_h f)(jh)}, \quad \mathbf{n}_j = ((D_h f)(jh), -1)/\omega_j, \quad \mathbf{s}_j = (1, (D_h f)(jh))/\omega_j, \tag{3.9}$$

so that $\mathbf{n}_j$ and $\mathbf{s}_j$ are approximations to $\mathbf{n}(\mathbf{x}_j)$ and $\mathbf{s}(\mathbf{x}_j)$. Furthermore, let $k_{ij} = k(ih, jh)$ and $\tilde{k}_{ij} = \tilde{k}(ih, jh)$ for $i, j \in \mathbb{Z}$. We define discrete operators $\bar{K}_N, \tilde{K}_N\colon l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$, related to the integral operator K, by

$$(\bar{K}_N\psi)_i = h\sum_{j\in\mathbb{Z}} k_{ij}\psi_j \quad \text{and} \quad (\tilde{K}_N\psi)_i = h\sum_{j\in\mathbb{Z}} \tilde{k}_{ij}\psi_j, \quad i \in \mathbb{Z},$$

and note that $(\bar{K}_N L_N\psi)_i = K_N\psi(ih)$, where $i \in \mathbb{Z}$, so that from (3.2) it follows that the sequence $\bar{\mu}_N := \{\mu_N(jh)\}_{j\in\mathbb{Z}}$ satisfies the equation

$$\bar{\mu}_N = \phi_N + \bar{K}_N\bar{\mu}_N.$$

The approximate Nyström scheme we are proposing is to solve, instead of this equation, the equation

$$\tilde{\mu}_N = \phi_N + \tilde{K}_N\tilde{\mu}_N. \tag{3.10}$$

We calculate $\tilde{\mu}_N = \{\tilde{\mu}_j\}_{j\in\mathbb{Z}}$ by solving (3.10), which is the infinite set of linear equations

$$\tilde{\mu}_i = \phi_i + h\sum_{j\in\mathbb{Z}} \tilde{k}_{ij}\tilde{\mu}_j, \quad i \in \mathbb{Z}. \tag{3.11}$$

The attraction of solving (3.11) in preference to (3.2) is that computing the coefficients $\tilde{k}_{ij}$ requires only the values of $f(ih)$, where $i \in \mathbb{Z}$, and not also the values of f' and f'' at all the grid points.

The next result on the existence and boundedness of $\bar{K}_N$ and $(I - \bar{K}_N)^{-1}$ follows from Theorems 3.2 and 3.3 by standard arguments for Nyström methods (see Atkinson, 1997, p. 113). In this (and subsequent) theorems we will use $\|\cdot\|_\infty$ to denote the induced operator norm for bounded operators on $l^\infty(\mathbb{Z})$.

THEOREM 3.9. If $f \in BC^3(\mathbb{R})$, $\|f\|_{BC^3(\mathbb{R})} \leqslant C_f$ and $C_f > 0$, then there exist $\bar{N} \in \mathbb{N}$ and $C > 0$ such that

$$\|\bar{K}_N\|_\infty \leqslant C \quad \text{for } N \in \mathbb{N}, \quad \text{and} \quad \|(I - \bar{K}_N)^{-1}\|_\infty \leqslant C \quad \text{for } N \geqslant \tilde{N},$$

where C depends only on n, $f_\pm$, H and C_f.

We now show the accuracy of $\tilde{K}_N$ as an approximation to $\bar{K}_N$.

THEOREM 3.10. If $f \in BC^{n+2}(\mathbb{R})$ and $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ for some $C_f > 0$ and $n \in \mathbb{N}_0$ and with n even, then there exists $C > 0$ such that

$$\|\bar{K}_N - \tilde{K}_N\|_\infty \leqslant C h^{n+1} \log(1+N) \quad \text{for } N \in \mathbb{N},$$

where C depends only on n, $f_\pm$, H and C_f.

Proof. From (2.3) and the definitions of k and $\tilde{k}$ we see that, for $i, j \in \mathbb{Z}$, where $i \neq j$, we have

$$k_{ij} - \tilde{k}_{ij} = -\frac{1}{2\pi}\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} - \frac{\mathbf{x}_i^r - \mathbf{x}_j}{|\mathbf{x}_i^r - \mathbf{x}_j|^2}\right) \cdot (\mathbf{n}(\mathbf{x}_j)\omega(jh) - \mathbf{n}_j\omega_j),$$

while, from (2.7) and (2.9), for $i = j$ we have

$$k_{ii} - \tilde{k}_{ii} = -\frac{1}{2\pi}\left(\frac{f''(ih)}{w(ih)^2} - \frac{D_h^2 f(ih)}{\omega_i^2}\right).$$

Lemma 2.1 of Preston *et al.* (2008) implies that

$$\left|\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} - \frac{\mathbf{x}_i^r - \mathbf{x}_j}{|\mathbf{x}_i^r - \mathbf{x}_j|^2}\right| \leqslant \frac{c}{(ih - jh)^2}, \quad i, j \in \mathbb{Z}, \quad i \neq j, \tag{3.12}$$

and clearly also

$$\left|\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} - \frac{\mathbf{x}_i^r - \mathbf{x}_j}{|\mathbf{x}_i^r - \mathbf{x}_j|^2}\right| \leqslant \frac{c}{|i - j|h}, \quad i, j \in \mathbb{Z}, \quad i \neq j, \tag{3.13}$$

where $c > 0$ depends only on $f_\pm$ and H. Combining these results with Theorem 3.7, we have

$$\begin{aligned}
\|\bar{K}_N - \tilde{K}_N\|_\infty &= \sup_{i\in\mathbb{Z}} h \sum_{j\in\mathbb{Z}} |k_{ij} - \tilde{k}_{ij}| \\
&\leqslant \sup_{i\in\mathbb{Z}} \frac{h}{2\pi}\left(\sum_{j\in\mathbb{Z}, j\neq i}\left|\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} - \frac{\mathbf{x}_i^r - \mathbf{x}_j}{|\mathbf{x}_i^r - \mathbf{x}_j|^2}\right| |\mathbf{n}(jh)w(jh) - \mathbf{n}_j w_j|\right. \\
&\qquad\qquad \left. + \left|\frac{f''(ih)}{w(ih)^2} - \frac{D_h^2 f(ih)}{w_i^2}\right|\right) \\
&\leqslant \sup_{i\in\mathbb{Z}} Ch\left(\sum_{j\in\mathbb{Z}, |i-j|\geqslant N} \frac{h^{n+1}}{(ih - jh)^2} + \sum_{j\in\mathbb{Z}, 1\leqslant|i-j|<N} \frac{h^{n+1}}{|i - j|h} + h^n\right)
\end{aligned}$$

$$\leqslant C\left(h^n \sum_{j=N}^{\infty} \frac{1}{j^2} + h^{n+1} \sum_{j=1}^{N-1} \frac{1}{j} + h^{n+1}\right)$$

$$\leqslant Ch^n(2h + h\log(1+N)) \leqslant Ch^{n+1}\log(1+N). \qquad \square$$

The following is a special case of a standard Banach algebra perturbation result (e.g., Rudin, 1991, p. 248).

THEOREM 3.11. If $(I - \bar{K}_N)^{-1}$ exists and is bounded, and

$$\|\bar{K}_N - \tilde{K}_N\|_\infty \leqslant \frac{1}{2\|(I - \bar{K}_N)^{-1}\|_\infty}, \tag{3.14}$$

then $(I - \tilde{K}_N)^{-1}$ exists and is bounded with the bound given by

$$\|(I - \tilde{K}_N)^{-1}\|_\infty \leqslant 2\|(I - \bar{K}_N)^{-1}\|_\infty. \tag{3.15}$$

We now present the main convergence result for the numerical scheme defined by (3.10).

THEOREM 3.12. If $f \in BC^{n+2}(\mathbb{R})$ and $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ for some $C_f > 0$ and $n \in \mathbb{N}_0$ with n even, then there exist $\tilde{N} \in \mathbb{N}$ and $C > 0$ such that, for all $N \geqslant \tilde{N}$, a uniquely determined solution $\tilde{\mu}_N \in l^\infty(\mathbb{Z})$ to (3.10) exists and, for $\phi_0 \in BC^n(\mathbb{R})$, we have

$$\|L_N\mu - \tilde{\mu}_N\|_\infty \leqslant C\|\phi_0\|_{BC^n(\mathbb{R})}h^n,$$

where C depends only on n, $f_\pm$, H and C_f.

Proof. By Theorems 3.3 and 3.10 we can choose $\tilde{N}$ such that, for all $N > \tilde{N}$, (3.3) and (3.14) hold and therefore, by Theorem 3.11, $(I - \tilde{K}_N)^{-1}$ exists and is bounded by (3.15). So, for $N > \tilde{N}$, (3.10) has a unique solution $\tilde{\mu}_N = (I - \tilde{K}_N)^{-1}\phi_N$. Furthermore, from Theorem 3.9, $\bar{\mu} = (I - \bar{K}_N)^{-1}\phi_N$. Combining these relationships, we have

$$\begin{aligned}
\bar{\mu}_N - \tilde{\mu}_N &= (I - \tilde{K}_N)^{-1}(I - \tilde{K}_N)\bar{\mu}_N - (I - \tilde{K}_N)^{-1}\phi_N \\
&= (I - \tilde{K}_N)^{-1}(I - \tilde{K}_N)\bar{\mu}_N - (I - \tilde{K}_N)^{-1}(I - \bar{K}_N)\bar{\mu}_N \\
&= (I - \tilde{K}_N)^{-1}(\bar{K}_N - \tilde{K}_N)\bar{\mu}_N
\end{aligned}$$

and therefore, by Theorems 3.3, 3.4 and 3.10, we have

$$\begin{aligned}
\|L_N\mu - \tilde{\mu}_N\|_\infty &\leqslant \|L_N\mu - \bar{\mu}_N\|_\infty + \|\bar{\mu}_N - \tilde{\mu}_N\|_\infty \\
&\leqslant \|\mu - \mu_N\|_{BC(\mathbb{R})} + \|(I - \bar{K}_N)^{-1}\|_\infty\|\bar{K}_N - \tilde{K}_N\|_\infty\|\bar{\mu}\|_{BC(\mathbb{R})} \\
&\leqslant C\|\phi_0\|_{BC^n(\mathbb{R})}h^n,
\end{aligned}$$

as required. $\square$

3.4 *Velocity approximation*

We now analyse an approximation to the velocity $\boldsymbol{\nu}$, given by (2.18), by utilizing the discrete derivative operator and $\tilde{\mu}_N$, given by (3.10), in an approximation to M. Precisely, we will approximate velocity values on a uniform grid, i.e., approximate $\boldsymbol{\nu}_N := L_N \boldsymbol{\nu} = \{\boldsymbol{\nu}_{j,N}\}_{j\in\mathbb{Z}}$.

To construct a first approximation, for $i, j \in \mathbb{Z}$ and $\psi \in BC(\mathbb{R})$ let $m_{ij}(\psi) = m[\psi](ih, jh)$, where m is given by (2.16). We define an operator $\bar{M}_N \colon BC(\mathbb{R}) \to l^\infty(\mathbb{Z})$ by

$$(\bar{M}_N \psi)_i := h \sum_{j\in\mathbb{Z}} m_{ij}(\psi), \quad i \in \mathbb{Z},$$

so that $\bar{M}$ is a trapezium rule approximation to the operator $L_N M$, where M is defined by (2.17). Then a first approximation to $\boldsymbol{\nu}_N = \{\boldsymbol{\nu}_{j,N}\}_{j\in\mathbb{Z}}$ is $\bar{\boldsymbol{\nu}}_N := \{\bar{\boldsymbol{\nu}}_{j,N}\}_{j\in\mathbb{Z}}$, where

$$\bar{\boldsymbol{\nu}}_{j,N} = \frac{\phi'(jh)}{\omega(jh)}\mathbf{s}(jh) + (\bar{M}\mu)_j \mathbf{n}(jh), \quad j \in \mathbb{Z}. \tag{3.16}$$

LEMMA 3.13. If $f \in BC^{n+2}(\mathbb{R})$, $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ for some $C_f > 0$, and $\mu \in BC^{n+2}(\mathbb{R})$ solves (2.5), then

$$\max_{j\in\mathbb{Z}} |\boldsymbol{\nu}_{j,N} - \bar{\boldsymbol{\nu}}_{j,N}| \leqslant C\|\phi_0\|_{BC^n(\mathbb{R})} h^n,$$

where $C > 0$ depends only on n, $f_\pm$, H and C_f.

Proof. The only approximation in (3.16) is in the Dirichlet-to-Neumann operator M. By Preston *et al.* (2008, Lemma 2.1) (see (3.12)), Theorem 2.5 and (2.16), we see that $m[\mu](\tau, \cdot) \in BC^n_p(\mathbb{R})$, where $p \geqslant 2$ and $\tau \in \mathbb{R}$. Therefore, by Lemma 3.1, we have

$$|(M\mu)(\tau) - I_h m[\mu](\tau, \cdot)| \leqslant Ch^n.$$

□

We next construct a fully discrete approximation to $\boldsymbol{\nu}_N$, using Lemma 3.13 to analyse its accuracy. Recalling the approximations ω_j, $\mathbf{n}_j$ and $\mathbf{s}_j$ introduced in (3.9) and writing $\mathbf{x}_j$ and $\mathbf{n}_j$ in terms of their components as $\mathbf{x}_j = (x_{j,1}, x_{j,2})$ and $\mathbf{n}_j = (n_{j,1}, n_{j,2})$, we define $\tilde{m}_{ij} \colon l^\infty(\mathbb{Z})^3 \to l^\infty(\mathbb{Z}^2)$ by

$$\begin{aligned}
&\tilde{m}_{ij}(\{\psi_k\}_{k\in\mathbb{Z}}, \{\psi'_k\}_{k\in\mathbb{Z}}, \{\psi''_k\}_{k\in\mathbb{Z}}) \\
&= -\frac{1}{2\pi}\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} - \frac{\mathbf{x}^r_i - \mathbf{x}_j}{|\mathbf{x}^r_i - \mathbf{x}_j|^2}\right) \cdot \left((\mathbf{n}_j \mathbf{n}_i \cdot \mathbf{s}_j + \mathbf{s}_j \mathbf{n}_i \cdot \mathbf{n}_j)\frac{\psi'_j}{\omega_j} - \mathbf{s}_j \frac{\psi'_i}{\omega_i}\right) \\
&\quad + \frac{1}{\pi}\left(\frac{(2(x^r_{i,1} - x_{j,1})(x^r_{i,2} - x_{j,2}), |\mathbf{x}^r_i - \mathbf{x}_j|^2)}{|\mathbf{x}^r_i - \mathbf{x}_j|^4}\right) \cdot ((\mathbf{n}_i n_{j,2} - \mathbf{s}_i n_{j,1})\omega_j \psi_j) \\
&\quad + \frac{1}{\pi}\left(\frac{x^r_{i,2} - x_{j,2}}{|\mathbf{x}_i - \mathbf{x}^r_j|^4}\right) n_{i,1} \psi'_j, \quad i \neq j,
\end{aligned}$$

and by

$$\tilde{m}_{ii}(\{\psi_k\}_{k\in\mathbb{Z}}, \{\psi'_k\}_{k\in\mathbb{Z}}, \{\psi''_k\}_{k\in\mathbb{Z}})$$
$$= \frac{1}{2\pi\omega_i^2}\left(\psi''_i - \frac{(D_h f)(ih)(D_h^2 f)(ih)\psi'_i}{\omega_i^2}\right) + \frac{1}{4H^2\pi}(\omega_i\psi_i + n_{i,1}\psi'_i), \quad i = j.$$

The point of this definition is that, where μ is the solution to the integral equation (2.6), $\tilde{m}_{ij}(L_N\mu, L_N\mu', L_N\mu'')$ is a first approximation of $m_{ij}(\mu)$ obtained by approximating the derivatives of f by the discrete derivative operator (3.6). Moreover, $\tilde{m}_{ij}(\tilde{\mu}_N, \tilde{D}_h\tilde{\mu}_N, \tilde{D}_h^2\tilde{\mu}_N)$ is a further fully discrete approximation, obtained by additionally approximating $L_N\mu$ by $\tilde{\mu}_N$, given by (3.11), and computing its numerical derivatives using (3.8). Using these approximations, we define the operators $\hat{M}_N\colon BC(\mathbb{R}) \to l^\infty(\mathbb{Z})$ and $\tilde{M}_N\colon l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$, which are approximations to $\bar{M}_N$ and $\bar{M}_N L_N$, respectively, by

$$(\hat{M}_N\mu)_i := h\sum_{j\in\mathbb{Z}} \tilde{m}_{ij}(L_N\mu, L_N\mu', L_N\mu''), \quad i \in \mathbb{Z},$$

and

$$(\tilde{M}_N\tilde{\mu}_N)_i := h\sum_{j\in\mathbb{Z}} \tilde{m}_{ij}(\tilde{\mu}_N, \tilde{D}_h\tilde{\mu}_N, \tilde{D}_h^2\tilde{\mu}_N), \quad i \in \mathbb{Z}. \tag{3.17}$$

Using $\tilde{M}_N\tilde{\mu}_N$, we define our final fully discrete velocity approximation $\tilde{\boldsymbol{\nu}}_N = \{\tilde{\boldsymbol{\nu}}_{j,N}\}_{j\in\mathbb{Z}}$ by

$$\tilde{\boldsymbol{\nu}}_{j,N} = \frac{(\tilde{D}_h L_N\phi)_j}{\omega_j}\mathbf{s}_j + (\tilde{M}_N\tilde{\mu}_N)_j\mathbf{n}_j, \quad j \in \mathbb{Z}. \tag{3.18}$$

In the last theorem of this paper we analyse the convergence of $\tilde{\boldsymbol{\nu}}_N$ to $\boldsymbol{\nu}_N$.

THEOREM 3.14. If $\phi_0 \in BC^n(\mathbb{R})$, $f \in BC^{n+2}(\mathbb{R})$ and $\|f\|_{BC^{n+2}(\mathbb{R})} \leqslant C_f$ for some $C_f > 0$ and some $n \in \mathbb{N}_0$ with n even, then there exists $C > 0$, depending only on n, $f_\pm$, H and C_f, such that

$$\max_{j\in\mathbb{Z}} |\boldsymbol{\nu}_{j,N} - \tilde{\boldsymbol{\nu}}_{j,N}| \leqslant C\|\phi_0\|_{BC^n(\mathbb{R})}h^{n-2}$$

for all $N \geqslant \tilde{N}$, where $\tilde{N}$ is as defined in Theorem 3.12.

Proof. We first note that solving the integral equation (2.6), with $\phi_0 \in BC^n(\mathbb{R})$ and $f \in BC^{n+2}(\mathbb{R})$, gives, by Theorem 2.3, $\mu \in BC^n(\mathbb{R})$ and hence, by Theorem 2.5, $m \in BC^{n-2}(\mathbb{R}^2)$. In the remainder of this proof we will show the accuracy of the approximations used in (3.18) to the five components (ϕ', ω, $\mathbf{n}$, $\mathbf{s}$ and M) of (2.18). By a straightforward application of Theorem 3.7 to $f' \in BC^{n+1}(\mathbb{R})$, and similarly to the analysis in Theorem 3.10, we have the three bounds

$$\max_{j\in\mathbb{Z}} |\omega(jh) - \omega_j| \leqslant C_1 h^{n+1}, \quad \max_{j\in\mathbb{Z}} |\mathbf{n}(jh) - \mathbf{n}_j| \leqslant C_2 h^{n+1}$$

and

$$\max_{j\in\mathbb{Z}} |\mathbf{s}(jh) - \mathbf{s}_j| \leqslant C_3 h^{n+1}, \tag{3.19}$$

where C_1, C_2 and C_3 depend only on n and C_f. Note that $(\tilde{D}_h L_N \phi)_j = (L_N D_h \phi)_j$ for $j \in \mathbb{Z}$, and that $\phi' \in BC^{n-1}(\mathbb{R})$. Therefore, by applying Theorem 3.7 we also have

$$\max_{j\in\mathbb{Z}} |\phi'(jh) - (\tilde{D}_h L_N \phi)_j| \leqslant C_4 h^{n-1},$$

where C_4 depends only on n and C_f.

All that remains is to prove the accuracy of $\tilde{M}_N \tilde{\mu}_N$ as an approximation to $M\mu$, and to do this we analyse the successive approximations given by (3.17). We have, by Lemma 3.13 with $\mu \in BC^n(\mathbb{R})$, that $\|L_N M\mu - \bar{M}_N \mu\|_\infty \leqslant Ch^{n-2}$. Furthermore, by (3.12), (3.13) and (3.19), we have

$$\begin{aligned}
\|\bar{M}_N \mu - \hat{M}_N \mu\|_\infty &= \sup_{i\in\mathbb{Z}} h \sum_{j\in\mathbb{Z}} |m_{ij}(\mu) - \tilde{m}_{ij}(L_N\mu, L_N\mu', L_N\mu'')| \\
&\leqslant Ch \left(\sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}, j\neq i} \frac{h^{n+1}}{(ih - jh)^2} + h^n \right) \\
&\leqslant Ch^n \left(\sum_{j\in\mathbb{N}} \frac{1}{j^2} + h \right) \leqslant Ch^n,
\end{aligned}$$

where C depends only on n, $f_\pm$, H and C_f.

Finally, by Theorem 3.12, $\|L_N\mu - \tilde{\mu}_N\|_\infty \leqslant Ch^n$ and therefore, by Theorem 3.8 with $p = n$,

$$\|L_N\mu' - \tilde{D}_h \tilde{\mu}\|_\infty \leqslant Ch^{n-2}, \quad \|L_N\mu'' - \tilde{D}_h^2 \tilde{\mu}\|_\infty \leqslant Ch^{n-3}.$$

Now, utilizing these bounds and (3.12), (3.13) and (3.19), we have

$$\begin{aligned}
\|\hat{M}_N \mu - \tilde{M}_N \tilde{\mu}\|_\infty &= \sup_{i\in\mathbb{Z}} h \sum_{j\in\mathbb{Z}} |\tilde{m}_{ij}(L_N\mu, L_N\mu', L_N\mu'') - \tilde{m}_{ij}(\tilde{\mu}_N, \tilde{D}_h\tilde{\mu}_N, \tilde{D}_h^2\tilde{\mu}_N)| \\
&\leqslant Ch \left(h^{n-3} + \sum_{|j|\leqslant N, j\neq 0} h^{n-2} + \sum_{|j|\geqslant N} \frac{h^{n-2}}{(jh)^2} \right) \\
&\leqslant Ch^{n-2} \left(1 + hN + \frac{1}{hN} \right) \leqslant Ch^{n-2},
\end{aligned}$$

where C depends only on n, $f_\pm$, H and C_f, as required. □

4. Numerical results

In this final section we give numerical results that illustrate the proven convergence rates. To produce these numerical results we first reduce the infinite system (3.11) to a finite linear system, doing this by one of two methods.

The first method is a basic truncation scheme that corresponds to replacing the range of integration of $\mathbb{R}$ in (2.4) and (2.5) by the finite interval $[-A, A]$, where $A = N_A h$, for some $N_A \in \mathbb{N}$. Precisely, the numerical scheme is to compute an approximation to μ on $[-A, A]$ by solving (3.11) with the range of summation reduced from $\mathbb{Z}$ to $\{-N_A, \ldots, N_A\}$, i.e., by solving

$$\tilde{\mu}_i = \phi_i + h \sum_{j=-N_A}^{N_A} \tilde{k}_{ij} \tilde{\mu}_j, \quad i = -N_A, \ldots, N_A. \tag{4.1}$$

Then an approximation to ϕ is given by (2.4) approximated by the trapezium rule, that is,

$$\phi(x) \approx \frac{-h}{2\pi} \sum_{j=-N_A}^{N_A} \left(\frac{(x-\mathbf{x}_j)\cdot \mathbf{n}_j}{(x-\mathbf{x}_j)^2} - \frac{(x-\mathbf{x}_j^r)\cdot \mathbf{n}_j}{(x-\mathbf{x}_j^r)^2} \right) w_j \tilde{\mu}_j, \quad x \in \Omega. \tag{4.2}$$

Moreover, to approximate the velocity on Γ we use (3.17) and (3.18) with the range of summation in (3.17) reduced to $\{-N_A, \ldots, N_A\}$. In this paper we do not make an attempt to analyse the additional errors introduced by these truncations of the range of summation or their stability and convergence. We note that, in Meier (2001), Meier & Chandler-Wilde (2001), Chandler-Wilde *et al.* (2002), Haseloh (2004) and Lindner (2006), this truncation process, a so-called 'finite-section' approximation, was studied in detail for several related problems.

In the second method we achieve a finite linear system by assuming (or approximating by) a periodic boundary and periodic boundary potential, thus enabling the infinite system of equations to be reduced to a finite system over a single period. Many other works, for example, Dold (1992), Beale *et al.* (1996) and Baker & Beale (2004), have shown results for this periodic case. To determine the discrete periodic system we must first reformulate the infinite system given by (3.11). As the boundary and potential are periodic, it follows, from the compactness of the operator $\bar{K}_N$ in (3.10) on the space of bounded periodic sequences and the Fredholm alternative, that the solution μ_N of (3.10) is also periodic. Thus, fixing on the case that the boundary and boundary data are periodic with period 2π, then f_j, ϕ_j and μ_j all share the periodicity that $a_j = a_{j+mN}$ for $m \in \mathbb{Z}$, as do the dependent variables w_j, $\mathbf{n}_j$ and $\mathbf{s}_j$. Taking advantage of this periodicity, we can rewrite (3.10) as

$$\begin{aligned}\tilde{\mu}_i = \phi_i + h\frac{D_h^2 f(ih)}{w_i}\tilde{\mu}_i - h\sum_{j=1, j\neq i}^{N} \tilde{\mu}_j w_j \mathbf{n}_i \cdot \sum_{k=-\infty}^{\infty} \nabla_x \Phi(x, \mathbf{x}_{j+kN})\Big|_{x=\mathbf{x}_i} \\ - h\sum_{j=1}^{N} \tilde{\mu}_j w_j \mathbf{n}_i \cdot \sum_{k=-\infty}^{\infty} \nabla_x \Phi(x, \mathbf{x}^r_{j+kN})\Big|_{x=\mathbf{x}_i}, \quad i = 1, \ldots, N.\end{aligned} \tag{4.3}$$

It is convenient at this point to use the isomorphism of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, thinking of $\mathbf{x}_j = (x_{j,1}, x_{j,2})$ and $\mathbf{n}_j = (n_{j,1}, n_{j,2})$ as points $\mathbf{x}_j = x_{j,1} + \mathrm{i}x_{j,2}$ and $\mathbf{n}_j = n_{j,1} + \mathrm{i}n_{j,2}$ in the complex plane. Then, applying equation (3.60) of Linton (1999), it follows that (4.3) can be written as

$$\begin{aligned}\tilde{\mu}_i = \phi_i + h\frac{D_h^2 f(ih)}{w_i}\tilde{\mu}_i - h\sum_{j=1, j\neq i}^{N} \mathrm{Re}\left(\bar{\mathbf{n}}_j \cot\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{2}\right)\right)\tilde{\mu}_j w_j \\ + h\sum_{j=1}^{N} \mathrm{Re}\left(\bar{\mathbf{n}}_j \cot\left(\frac{\mathbf{x}_i - \bar{\mathbf{x}}_j - 2H\mathrm{i}}{2}\right)\right)\tilde{\mu}_j w_j, \quad i = 1, \ldots, N,\end{aligned} \tag{4.4}$$

and we can apply similar formulae from Linton (1999) to obtain an analogous expression for the normal velocity approximation $\tilde{M}_N \tilde{\mu}_N$, starting from (3.17), and an analogous approximation for $\phi(x)$ as a finite sum, starting from (2.4) approximated by the trapezium rule with step length h.

In our numerical experiments we set $h = 2\pi/N$, with $N = 2, 4, 8, \ldots, 1024$, and choose $H = 1$ in the definition of Φ_H throughout. We construct examples for which we know the solution analytically by, having chosen a surface profile Γ, choosing a $\phi \in BC(\bar{\Omega}) \cap C^2(\Omega)$ that satisfies (1.2) in Ω. Clearly,

ϕ then satisfies the boundary value problem (1.5) with $\phi_0 := \phi|_\Gamma$, and we can compute analytically the normal velocity on Γ and the exact velocity potential at some test point in Ω. In the experiments below we choose as a test point $x = (0.1, -1.2)$.

In our first numerical example the surface Γ is sinusoidal, given by $\Gamma = \{(\sigma, 0.2\sin(\sigma)) : \sigma \in \mathbb{R}\}$, and the velocity potential is given by $\phi(x) = \Phi^{\mathrm{per}}_{\tilde{H}}(x, x^*)$, where $x \in \bar{\Omega}$, and where $\tilde{H} = 1.0$ and $x^* = (-0.2, 0.6)$. Here, for $H \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$, we have

$$\begin{aligned}\Phi^{\mathrm{per}}_H(x, y) &:= \sum_{k=-\infty}^{\infty} \Phi_H(x, y + 2\pi k\mathbf{e}_1)\\ &= \frac{1}{2}\left(\ln\left(2\left|\sin\left(\frac{\mathbf{x}-\mathbf{y}}{2}\right)\right|\right) - \ln\left(2\left|\sin\left(\frac{\mathbf{x}'-\mathbf{y}}{2}\right)\right|\right)\right)\end{aligned}$$

on using equation (3.60) of Linton (1999) again, where $\mathbf{x} = x_1 + x_2\mathrm{i}$ and $\mathbf{y} = y_1 + y_2\mathrm{i}$ are the points in the complex plane corresponding to x and y and $\mathbf{x}' = x_1 + (2H - x_2)\mathrm{i}$. The 2π-periodicity of Γ and of the Dirichlet data $\phi_0 := \phi|_\Gamma$ imply that the infinite linear system (3.11) reduces to the finite linear system (4.4). In Table 1 and Fig. 1 we tabulate and plot for this example two different relative errors as a function of N. The first of these is the relative error between the exact velocity potential at x, the test point, and the velocity potential calculated numerically. The second is the relative discrete ℓ_2 error between the known normal velocity and that computed numerically, that is, the relative ℓ_2 error in

TABLE 1 *Relative errors in potential at the test point and in normal velocity, plus values of EOC, for the first example (sinusoidal surface profile)*

N	Potential	Normal velocity
2	4.93×10^{-1}	7.25×10^{-1}
	2.62	0.44
4	8.02×10^{-2}	5.35×10^{-1}
	5.93	1.70
8	1.32×10^{-3}	1.65×10^{-1}
	6.97	2.46
16	1.05×10^{-5}	3.00×10^{-2}
	11.60	8.00
32	3.38×10^{-9}	1.17×10^{-4}
	25.20	10.13
64	8.77×10^{-17}	1.05×10^{-7}
	2.13	24.86
128	2.00×10^{-17}	3.44×10^{-15}
	0.03	2.27
256	1.96×10^{-17}	7.13×10^{-16}
	0.01	-1.29
512	1.95×10^{-17}	1.74×10^{-15}
	0.03	-1.38
1024	1.91×10^{-17}	4.53×10^{-15}

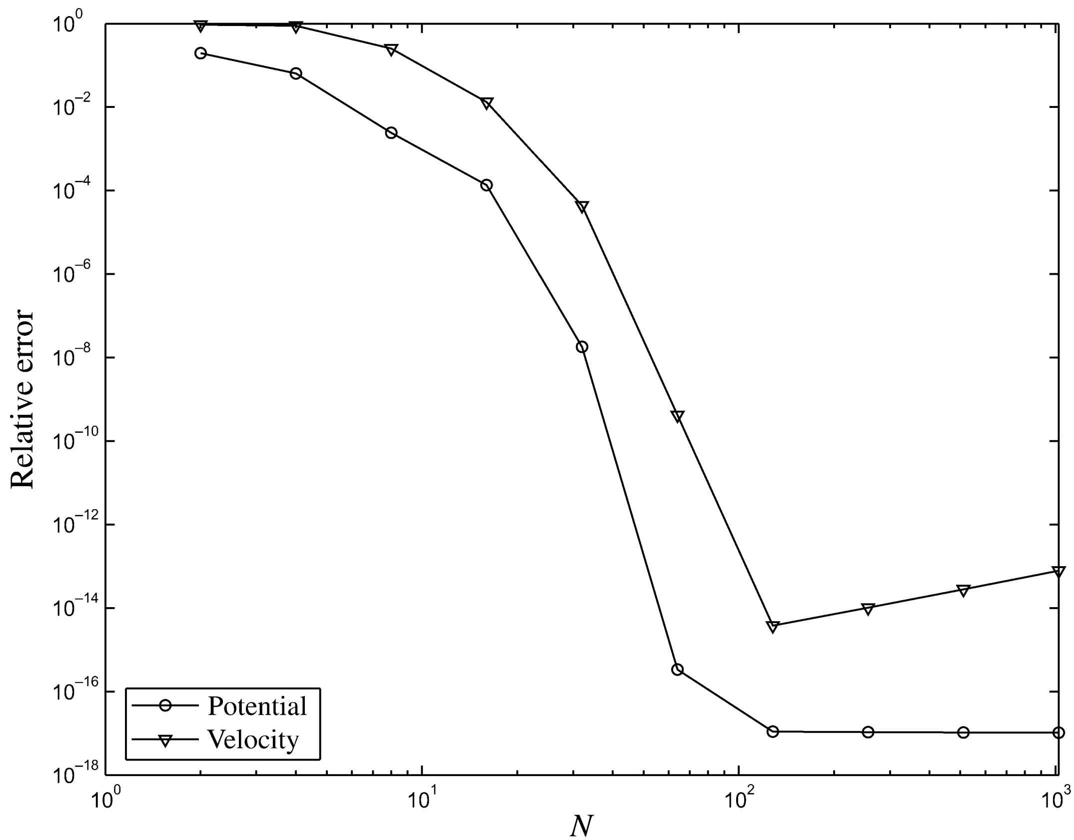

FIG. 1. Relative errors in potential at the test point and in normal velocity for the first example (sinusoidal surface profile).

the values $(\tilde{M}_N \tilde{\mu}_N)_i$ sampled at $i = 1, \ldots, N$, i.e., over one period. Estimated orders of convergence (EOC) are also tabulated, computed by the formula

$$\mathrm{EOC} = \log_2(\text{Error for given } N/\text{Error for } 2N),$$

so that $\mathrm{EOC} = p$ if the error is proportional to N^{-p}.

The numerical results in Table 1 and Fig. 1 are consistent with the superalgebraic convergence predicted by Theorems 3.12 and 3.14 when $f \in BC^{\infty}(\mathbb{R})$. Precisely, it can be seen that both approximations converge at an increasingly rapid rate, the values for EOC increasing, reaching a maximum value of over 20 before any further increase in accuracy is limited by rounding errors.

In the above example we have demonstrated, indirectly, the convergence predicted by Theorem 3.12 but have not shown this convergence directly since, for the above example, we do know the true density μ. In a second example, we consider the special case where the surface is flat, that is, $\Gamma := \{(\sigma, 0) : \sigma \in \mathbb{R}\}$. Choosing Dirichlet data $\phi_0 := \phi|_{\Gamma}$, where the velocity potential ϕ is given by

$$\phi(x) = -\frac{1}{2}(\Phi_H^{\mathrm{per}}(x, x^*) - \Phi_{3H}^{\mathrm{per}}(x, x^{**})), \quad x \in \bar{\Omega},$$

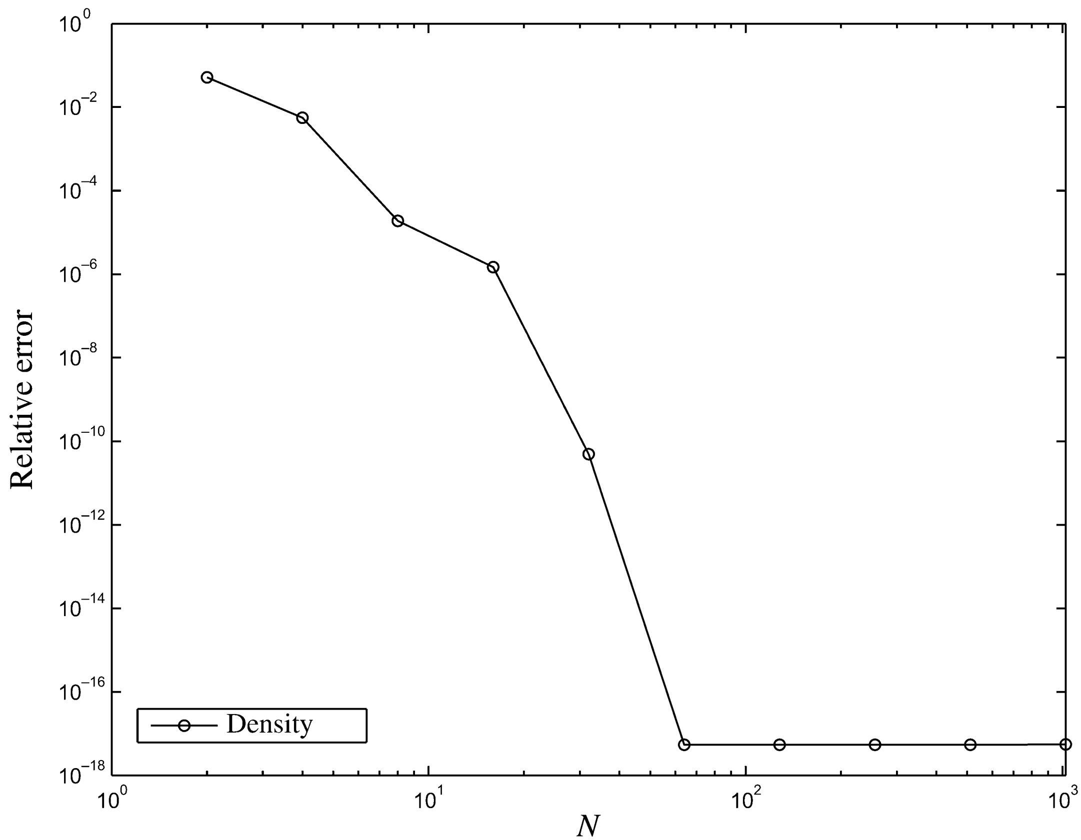

FIG. 2. Relative ℓ_2 error in density for the second example (flat surface).

where $x^* = (x_1^*, x_2^*)$, with $0 < x_2^* < H$ and $x^{**} = (x_1^*, 4H - x_2^*)$, it follows from Preston *et al.* (2008, Theorem 4.3.1) that the density μ in (2.4) and (2.5) is given by

$$\mu(\sigma) = \Phi_H^{\mathrm{per}}((\sigma, 0), x^*), \quad \sigma \in \mathbb{R}. \tag{4.5}$$

As in the first example, the 2π-periodicity of Γ and of the Dirichlet data $\phi_0 := \phi|_\Gamma$ imply that the infinite linear system (3.11) reduces to the finite linear system (4.4). In Fig. 2 we plot the relative discrete ℓ_2 error between the known density μ and its numerical approximation found by solving (3.11). The numerical results plotted in this figure illustrate the superalgebraic convergence predicted by Theorem 3.12 when $f \in BC^\infty(\mathbb{R})$.

In our third and final example we obtain a finite linear system by truncation (so that we use (4.1) and (4.2)), the boundary Γ has the Gaussian profile $\Gamma = \{(\sigma, 0.2\exp(-\sigma^2))\colon \sigma \in \mathbb{R}\}$ and the boundary data are $\phi_0 := \phi|_\Gamma$, where the potential ϕ is given by $\phi(x) = \Phi_H(x, x^*)$, where $x \in \bar{\Omega}$ and $x^* = (-0.2, 0.6)$. The truncation is performed with $A = P\pi$, where $P = 1, 2, 4, \ldots, 64$. We present the relative error between the exact velocity potential at x, the test point, and the velocity potential calculated numerically using (4.2) in Table 2 and, for each fixed P, values of EOC are also tabulated. It can be seen

TABLE 2 *Relative error in the approximation (4.2) to the potential at the test point and values of EOC for the third example (Gaussian surface profile)*

	P						
N	1	2	4	8	16	32	64
2	7.28×10^{-1}	7.29×10^{-1}	7.28×10^{-1}	7.28×10^{-1}	7.28×10^{-1}	7.28×10^{-1}	7.28×10^{-1}
	2.29	2.27	2.27	2.27	2.27	2.27	2.27
4	1.49×10^{-1}	1.51×10^{-1}	1.51×10^{-1}	1.51×10^{-1}	1.51×10^{-1}	1.51×10^{-1}	1.51×10^{-1}
	5.97	5.11	5.07	5.06	5.06	5.06	5.06
8	2.38×10^{-3}	4.38×10^{-3}	4.51×10^{-3}	4.52×10^{-3}	4.53×10^{-3}	4.53×10^{-3}	4.53×10^{-3}
	0.24	4.80	7.33	8.61	8.92	8.96	8.97
16	2.02×10^{-3}	1.57×10^{-4}	2.81×10^{-5}	1.15×10^{-5}	9.37×10^{-6}	9.09×10^{-6}	9.05×10^{-6}
	0.03	0.09	0.57	2.25	5.24	7.99	6.97
32	1.97×10^{-3}	1.47×10^{-4}	1.90×10^{-5}	2.43×10^{-6}	2.48×10^{-7}	3.57×10^{-8}	7.21×10^{-8}
	0.01	−0.00	−0.01	−0.04	−0.39	−0.18	4.11
64	1.96×10^{-3}	1.47×10^{-4}	1.90×10^{-5}	2.50×10^{-6}	3.24×10^{-7}	4.06×10^{-8}	4.17×10^{-9}
	0.00	0.00	−0.00	−0.00	−0.00	−0.04	−0.34
128	1.96×10^{-3}	1.47×10^{-4}	1.90×10^{-5}	2.50×10^{-6}	3.25×10^{-7}	4.17×10^{-8}	5.29×10^{-9}
	0.00	0.00	0.00	−0.00	−0.00	−0.00	
256	1.96×10^{-3}	1.47×10^{-4}	1.90×10^{-5}	2.50×10^{-6}	3.25×10^{-7}	4.17×10^{-8}	—
	0.00	0.00	0.00	−0.00	−0.00		
512	1.96×10^{-3}	1.47×10^{-4}	1.90×10^{-5}	2.50×10^{-6}	3.25×10^{-7}	—	—
	0.00	0.00	0.00	−0.00			
1024	1.96×10^{-3}	1.47×10^{-4}	1.90×10^{-5}	2.50×10^{-6}	—	—	—

TABLE 3 *Relative ℓ_2 error in normal velocity (with EOC) for the third example (Gaussian surface profile)*

	P						
N	1	2	4	8	16	32	64
2	6.75×10^{-1}	6.73×10^{-1}	6.73×10^{-1}	6.73×10^{-1}	6.73×10^{-1}	6.73×10^{-1}	6.73×10^{-1}
	0.25	0.21	0.21	0.21	0.21	0.21	0.21
4	5.68×10^{-1}	5.81×10^{-1}	5.81×10^{-1}	5.80×10^{-1}	5.80×10^{-1}	5.80×10^{-1}	5.80×10^{-1}
	1.12	1.12	1.12	1.12	1.12	1.12	1.12
8	2.61×10^{-1}	2.67×10^{-1}	2.66×10^{-1}	2.66×10^{-1}	2.66×10^{-1}	2.66×10^{-1}	2.66×10^{-1}
	3.37	3.34	3.34	3.34	3.34	3.34	3.34
16	2.52×10^{-2}	2.63×10^{-2}	2.63×10^{-2}	2.63×10^{-2}	2.63×10^{-2}	2.63×10^{-2}	2.63×10^{-2}
	1.64	4.15	4.18	4.18	4.18	4.18	4.18
32	8.13×10^{-3}	1.48×10^{-3}	1.45×10^{-3}	1.45×10^{-3}	1.45×10^{-3}	1.45×10^{-3}	1.45×10^{-3}
	−0.18	2.10	4.11	5.04	5.30	5.33	5.34
64	9.21×10^{-3}	3.46×10^{-4}	8.42×10^{-5}	4.42×10^{-5}	3.69×10^{-5}	3.60×10^{-5}	3.59×10^{-5}
	−0.14	−0.12	0.13	0.78	2.16	3.86	5.43
128	1.02×10^{-2}	3.76×10^{-4}	7.70×10^{-5}	2.57×10^{-5}	8.23×10^{-6}	2.48×10^{-6}	8.34×10^{-7}
	−0.10	−0.08	−0.01	−0.00	0.00	0.03	
256	1.09×10^{-2}	3.96×10^{-4}	7.76×10^{-5}	2.57×10^{-5}	8.22×10^{-6}	2.44×10^{-6}	—
	−0.06	−0.05	−0.01	−0.00	−0.00		
512	1.14×10^{-2}	4.09×10^{-4}	7.79×10^{-5}	2.57×10^{-5}	8.23×10^{-6}	—	—
	−0.04	−0.03	−0.00	−0.00			
1024	1.17×10^{-2}	4.18×10^{-4}	7.81×10^{-5}	2.57×10^{-5}	—	—	—

that the approximation given by (4.2) converges to $\phi(x)$ as $N \to \infty$ and $P \to \infty$ and that, for a fixed large P (when the errors induced by the truncation are small), the values for EOC increase initially as N increases up to nearly EOC $= 9$, consistent with the superalgebraic convergence predicted by Theorems 3.12 and 3.14 when $f \in BC^{\infty}(\mathbb{R})$. The relative errors from Table 2 are plotted in Fig. 3, where, for P large, the predicted superalgebraic convergence as N increases can be observed and, for N large, algebraic convergence as P increases can be observed. In Table 3 the relative ℓ_2 error between the known normal velocity and that calculated by reducing the range of summation to $\{-N_A, \dots, N_A\}$ in (3.17) is tabulated, that is, the discrete ℓ_2 error based on comparing $(\tilde{M}_N \tilde{\mu}_N)_i$ with the exact normal velocity for $i = -PN/2, \dots, PN/2$. The same values are plotted in Fig. 4. The trends are similar to those observed in Table 2 and Fig. 3, except that, for the same values of N, the relative errors are larger and the EOC values are not so large for the normal velocity. Furthermore, as P increases with N fixed and large, algebraic convergence is observed in Fig. 4 but at a slower rate than for the potential.

Further numerical results can be found in Preston (2007, Chapter 4).

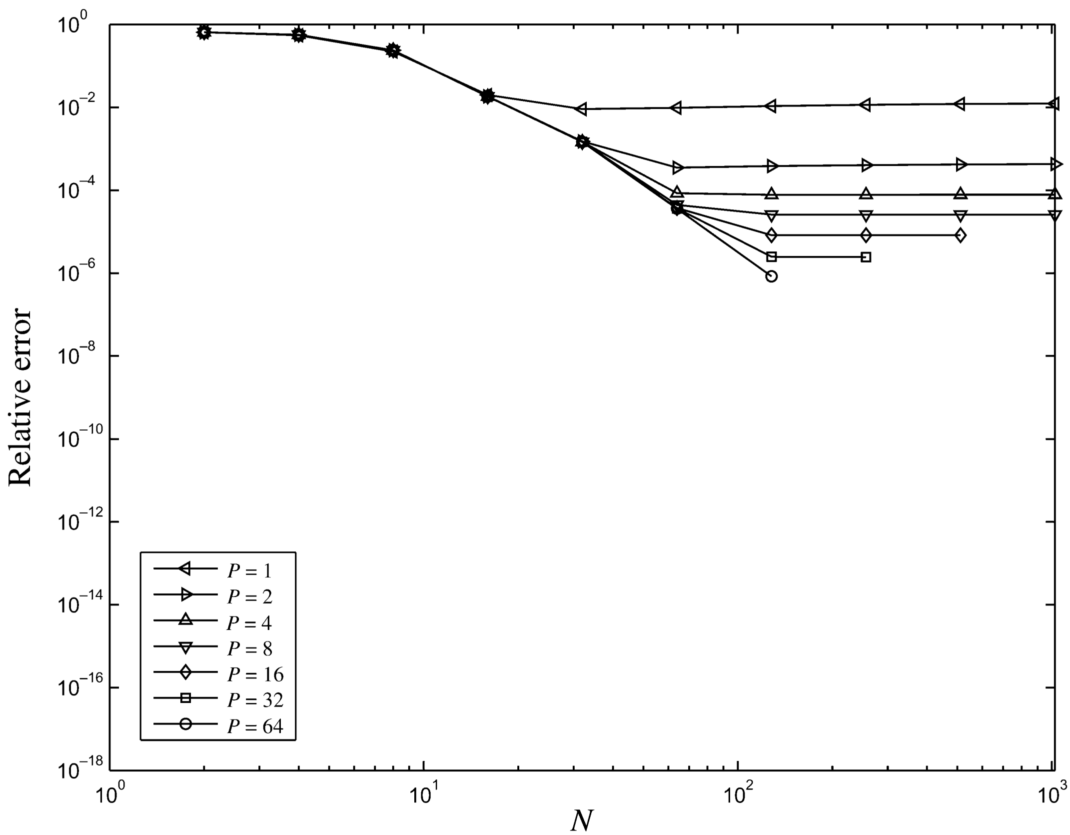

FIG. 3. Relative error in the approximation (4.2) to the potential at the test point for the third example (Gaussian surface profile).

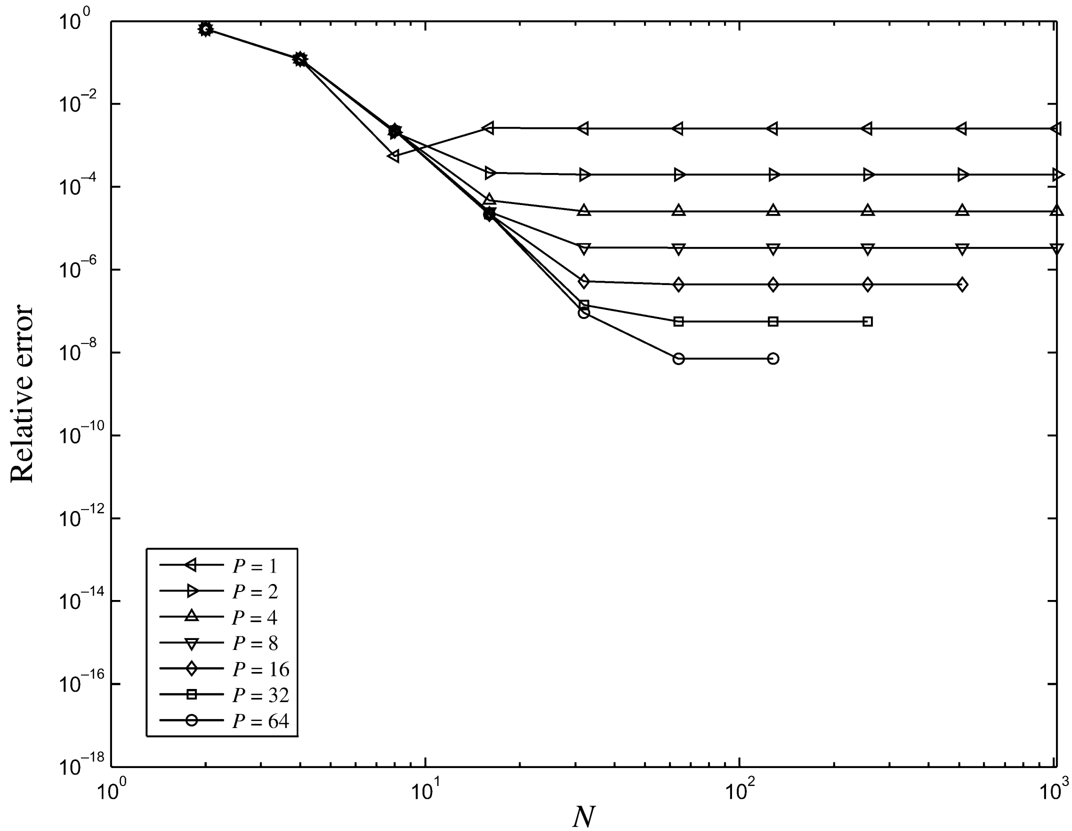

FIG. 4. Relative ℓ_2 error in normal velocity for the third example (Gaussian surface profile).

REFERENCES

ATKINSON, K. E. (1997) *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge, UK: Cambridge University Press.

BAKER, G. R. & BEALE, J. T. (2004) Vortex blob methods applied to interfacial motion. *J. Comput. Phys.*, **196**, 233–258.

BEALE, J. T., HOU, T. Y. & LOWENGRUB, J. (1996) Convergence of a boundary integral method for water waves. *SIAM J. Numer. Anal.*, **33**, 1797–1843.

CHANDLER-WILDE, S. N., RAHMAN, M. & ROSS, C. R. (2002) A fast two-grid and finite section method for a class of integral equations on the real line with application to an acoustic scattering problem in the half-plane. *Numer. Math.*, **93**, 1–51.

DOLD, J. (1992) An efficient surface-integral algorithm applied to unsteady gravity waves. *J. Comput. Phys.*, **103**, 90–115.

FOCHESATO, C. & DIAS, F. (2006) A fast method for nonlinear three-dimensional free-surface waves. *Proc. R. Soc. A Math. Phys.*, **462**, 2715–2735.

GILBARG, D. & TRUDINGER, N. S. (1977) *Elliptic Partial Differential Equations of Second Order*. New York: Springer.

HARDY, G. H. (1958) *A Course of Pure Mathematics*. Cambridge, UK: Cambridge University Press.

HASELOH, K. O. (2004) Second kind integral equations on the real line: solvability and numerical analysis in weighted spaces. *Ph.D. Thesis*, Universität Hannover, Germany.

HOU, T. Y. & ZHANG, P. (2002) Convergence of a boundary integral method for 3-D water waves. *Discrete Contin. Dyn. Syst. Ser. B*, **2**, 1–34.

KRESS, R. (1999) *Linear Integral Equations*. New York: Springer.

LINDNER, M. (2006) *Infinite Matrices and their Finite Sections*. Basel, Switzerland: Birkhäuser.

LINTON, C. M. (1999) Rapidly convergent representations of Green's functions for Laplace's equation. *Proc. R. Soc. A*, **455**, 1767–1797.

MEIER, A. (2001) Boundary element methods for outdoor sound propagation. *Ph.D. Thesis*, Brunel University, Uxbridge, UK.

MEIER, A., ARENS, T., CHANDLER-WILDE, S. N. & KIRSCH, A. (2000) A Nyström method for a class of integral equations on the real line with applications to scattering by diffraction gratings and rough surfaces. *J. Int. Equ. Appl.*, **12**, 281–321.

MEIER, A. & CHANDLER-WILDE, S. N. (2001) On the stability and convergence of the finite section method for integral equation formulations of rough surface scattering. *Math. Methods Appl. Sci.*, **24**, 209–232.

MEINARDUS, G. (1967) *Approximation of Functions: Theory and Numerical Methods*. New York: Springer.

PRESTON, M. D. (2007) A boundary integral equation method for solving second kind integral equations arising in unsteady water wave problems. *Ph.D. Thesis*, University of Reading, UK.

PRESTON, M. D., CHAMBERLAIN, P. G. & CHANDLER-WILDE, S. N. (2008) A boundary integral equation method for a boundary value problem arising in unsteady water waves. *J. Int. Equ. Appl.*, **20**, 121–152.

RUDIN, W. (1991) *Functional Analysis*. New York: McGraw-Hill.

XU, L. & GUYENNE, P. (2009) Numerical simulation of three-dimensional nonlinear water waves. *J. Comput. Phys.*, **228**, 8446–8466.