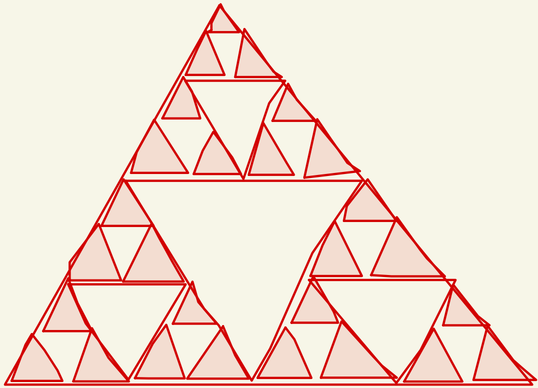
Sobolev Spaces, Integral Equations, and Scattering on non-Lipschitz and Fractal Sets

Simon Chandler-Wilde,

University of Reading



Part 3 Sobolev Spaces

Tempered Distributions

$C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$

space of rapidly decreasing C^∞ fns

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$\mathcal{S}^*(\mathbb{R}^n)$, dual space of $\mathcal{S}(\mathbb{R}^n)$ – "tempered distributions"

$\langle u, v\rangle \in \mathbb{C}$, for $u \in \mathcal{S}^*(\mathbb{R}^n)$, $v \in \mathcal{S}(\mathbb{R}^n)$ is action of linear functional u on v

Tempered Distributions

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space of rapidly decreasing C^∞ fns

$S^*(\mathbb{R}^n)$, dual space of $S(\mathbb{R}^n)$ – "tempered distributions"

$\langle u, v\rangle \in \mathbb{C}$, for $u \in S^*(\mathbb{R}^n)$, $v \in S(\mathbb{R}^n)$ is action of linear functional u on v

$\hat{u} = \mathcal{F}u$ is FT of u, $\mathcal{F}$ an isomorphism on $S(\mathbb{R}^n)$ and on $S^*(\mathbb{R}^n)$

$$\langle \hat{u}, v\rangle := \langle u, \hat{v}\rangle, \quad u \in S^*, v \in S$$

$\boxed{\text{Ex}}$ Given $z \in \mathbb{R}^n$ define

$\delta_z \in S^*(\mathbb{R}^n)$ by

$\langle \delta_z, v \rangle = v(z), \quad v \in S(\mathbb{R}^n)$

Show that $\mathcal{F}\delta_z = f$, where

$f(\xi) = (2\pi)^{-n/2} e^{-iz\xi}, \quad \xi \in \mathbb{R}^n$

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x)\,dx, \quad \xi \in \mathbb{R}^n, \quad u \in S(\mathbb{R}^n)$$

Sobolev spaces on $\mathbb{R}^n$

For $s \geq 0$, $H^s(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) \ \ \|u\|_{H^s} < \infty \}$

where

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi$$

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$$\|u\|_{H^0}^2 = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |u(x)|^2 dx = \|u\|_{L^2}^2$$

so $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$

Sobolev spaces on $\mathbb{R}^n$

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where

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi$$

$$\widehat{\partial_j u}(\xi) = i\xi_j \hat{u}(\xi) \Longrightarrow \|u\|_{H^1}^2 = \int_{\mathbb{R}^n} (1+\xi_1^2 + \quad + \xi_n^2) |\hat{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^n} \Big(|\hat{u}|^2 + \sum_j |\widehat{\partial_j u}|^2\Big)$$

$$= \int_{\mathbb{R}^n} \big(|u|^2 + |\nabla u|^2\big)$$

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Sobolev spaces on $\mathbb{R}^n$

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For $s>0$, $H^{-s}(\mathbb{R}^n) = \left(H^s(\mathbb{R}^n)\right)^*$

For $s>0$, $S(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$, so $H^{-s}(\mathbb{R}^n) \subset S^*(\mathbb{R}^n)$,

Sobolev spaces on $\mathbb{R}^n$

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For $s>0$, $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^*$

For $s>0$, $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$, so $H^{-s}(\mathbb{R}^n) \subset \mathcal{S}^*(\mathbb{R}^n)$,
indeed
$$H^{-s}(\mathbb{R}^n) = \{u \in \mathcal{S}^*(\mathbb{R}^n) \ \ \|u\|_{H^{-s}} < \infty\}$$

Proof just uses Riesz rep thm on $L^2(\mathbb{R}^n)$

Sobolev spaces on $\mathbb{R}^n$

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For $s > 0$, $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^*$

$$= \{u \in S^*(\mathbb{R}^n) \ \ \|u\|_{H^{-s}} < \infty\}$$

$\boxed{\text{Ex}}$ Show $\delta_z \in H^{-s}(\mathbb{R}^n)$

$\iff s > n/2$

Closed subspaces of $H^s(\mathbb{R}^n)$

For $\boxed{\text{open}}$ $\Omega \subset \mathbb{R}^n$,

$$\tilde{H}^s(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\mathbb{R}^n)}$$

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$$H^s_F := \{ u \in H^s(\mathbb{R}^n) . \operatorname{supp}(u) \subset F \}$$

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NB $u \in H^s_F \iff \langle u, v\rangle = 0, \ \forall v \in C_0^\infty(\mathbb{R}^n \setminus F)$

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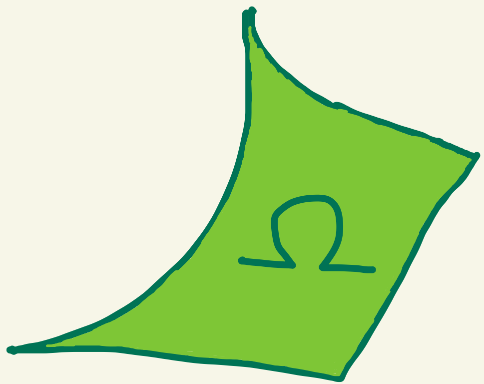
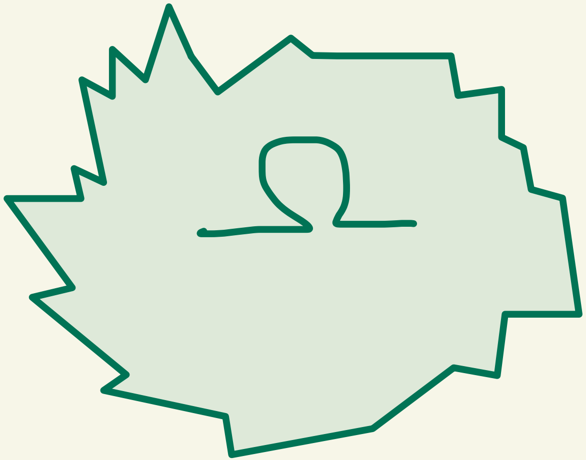
$\tilde{H}^s(\Omega) \subset H^s_{\overline{\Omega}}$ – often $\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$

$\boxed{\text{Thm}}$ $\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ if

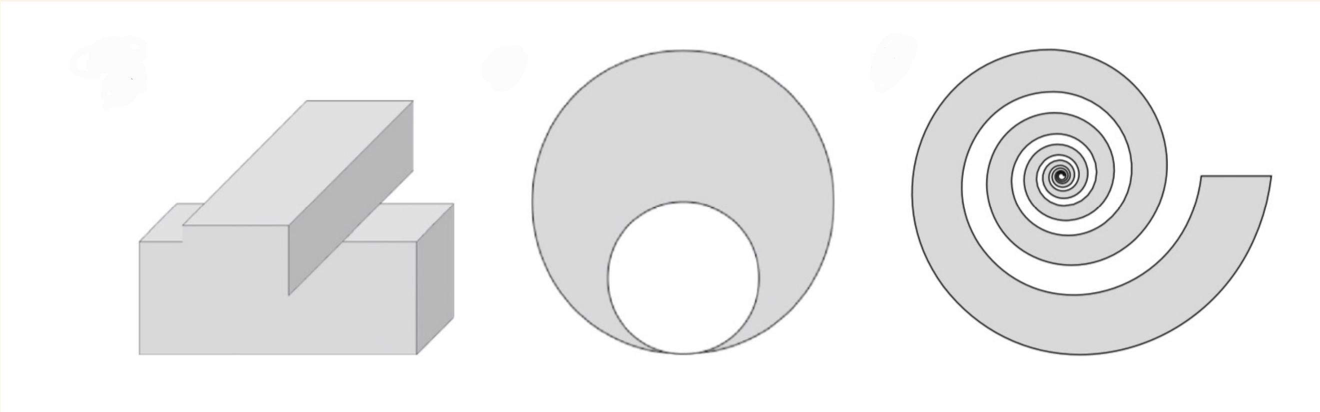
(i) Ω is C^0
and $s \in \mathbb{R}$





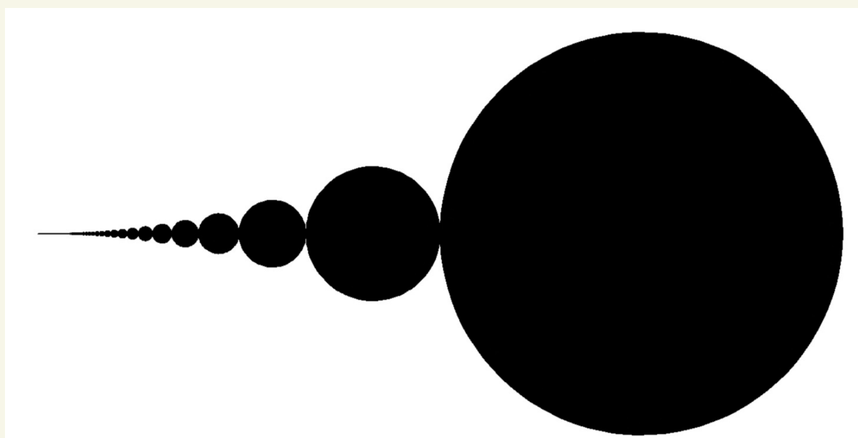
$\boxed{\text{Thm}}$ $\tilde{H}^s(\Omega) = H^s_{\bar{\Omega}}$ if

(ii) Ω is C^0 except at a finite number of points on $\partial\Omega$ and $|s| \leq \frac{1}{2}$ $(n=1)$, $|s| \leq 1$ $(n \geq 2)$



$\boxed{\text{Thm}}$ $\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ if

(iii) Ω is C^0 except at a countable set of points on $\partial\Omega$ with finitely many limit points and $|s| \leq \frac{1}{2}$ $(n=1)$, $|s| \leq 1$ $(n \geq 2)$

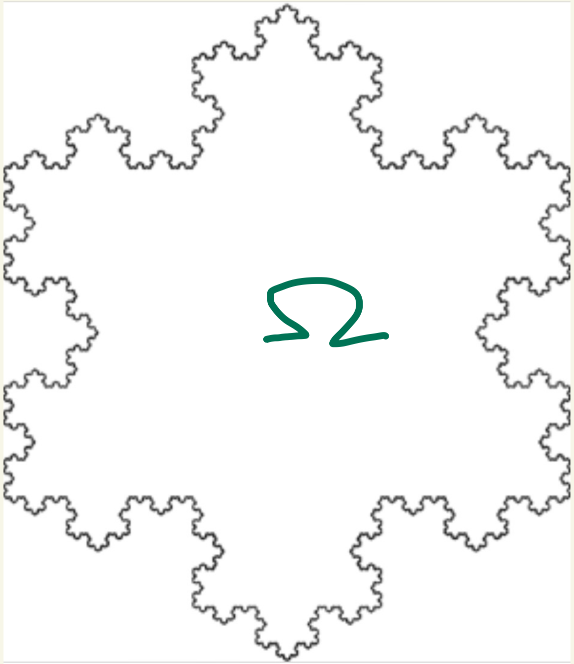


(C-W, Hewett, Moiola, IEOT 2017)

Thm $\widetilde{H}^s(\Omega) = H^s_{\overline{\Omega}}$ if

(iii) Ω is in a specific set of domains with fractal boundary, $s \in \mathbb{R}$

(Caetano, Hewett, Moiola 2019)



$\boxed{\text{Thm}}$ If $\Omega \subset \mathbb{R}^n$ open and $\overline{\Omega} = \mathbb{R}^n$ (so $H^s_{\overline{\Omega}} = H^s(\mathbb{R}^n)$) then

$$\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}} \iff H^{-s}_{\partial\Omega} = \{0\}$$

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Ex If $z \in \mathbb{R}^n$ and $\Omega = \mathbb{R}^n \setminus \{z\}$, so $\partial\Omega = \{z\}$,

$$\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}} \iff H^{-s}_{\partial\Omega} = \{0\} \iff s \leq \frac{n}{2}$$

Thm If $\Omega \subset \mathbb{R}^n$ open and $\overline{\Omega} = \mathbb{R}^n$ (so $H^s_{\overline{\Omega}} = H^s(\mathbb{R}^n)$) then

$$\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}} \Leftrightarrow H^{-s}_{\partial\Omega} = \{0\}$$

Proof $\Rightarrow$ Suppose $\phi \in H^{-s}_{\partial\Omega}$ & $\tilde{H}^s(\Omega) = H^s(\mathbb{R}^n)$. Then

$\forall\, \psi \in H^s(\mathbb{R}^n)$, $\langle \phi, \psi \rangle = \lim_{n\to\infty} \underbrace{\langle \phi, \underbrace{\psi_n}_{\in C_0^\infty(\Omega)} \rangle}_{=0}$

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Proof $\Leftarrow$ Exercise!

(Maz'ya, "Sobolev Spaces with ", Thm 13 2 1)

$\boxed{\text{Thm}}$ If $\Omega \subset \mathbb{R}^n$ open and $\overline{\Omega} = \mathbb{R}^n$ (so $H^s_{\overline{\Omega}} = H^s(\mathbb{R}^n)$) then

$$\tilde{H}^s(\Omega) = H^s_{\overline{\Omega}} \iff H^{-s}_{\partial\Omega} = \{0\}$$

$$\Longrightarrow \dim_H \partial\Omega < n - 2s$$

$\left(\frac{n}{2} \geq s > 0\right)$

Hausdorff dim $\in [0, n]$

Open problem

Necessary and sufficient conditions for

$$H^s(\Omega) = H^s_{\overline{\Omega}}$$

for general open $\Omega \subset \mathbb{R}^n$, $s \in \mathbb{R}$

(Review of what known in C-W, Hewett, Moiola, IEOT, 2017)

Open problem

True $\forall$ Jones domains?

Necessary and sufficient conditions for $H^s(\Omega) = H^s_{\overline{\Omega}}$ for general open $\Omega \subset \mathbb{R}^n$, $s \in \mathbb{R}$

(Review of what known in C-W, Hewett, Moiola, IEOT, 2017)

Sequences of subspaces and Mosco convergence

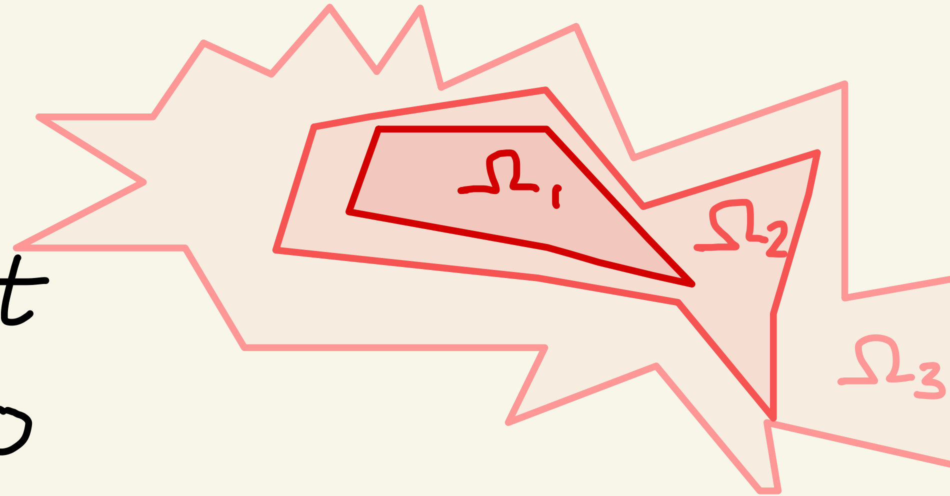
Suppose

$\Omega_1 \subset \Omega_2 \subset$

are open and let

$V_j := \widetilde{H}^s(\Omega_j)$ so

$V_1 \subset V_2 \subset$



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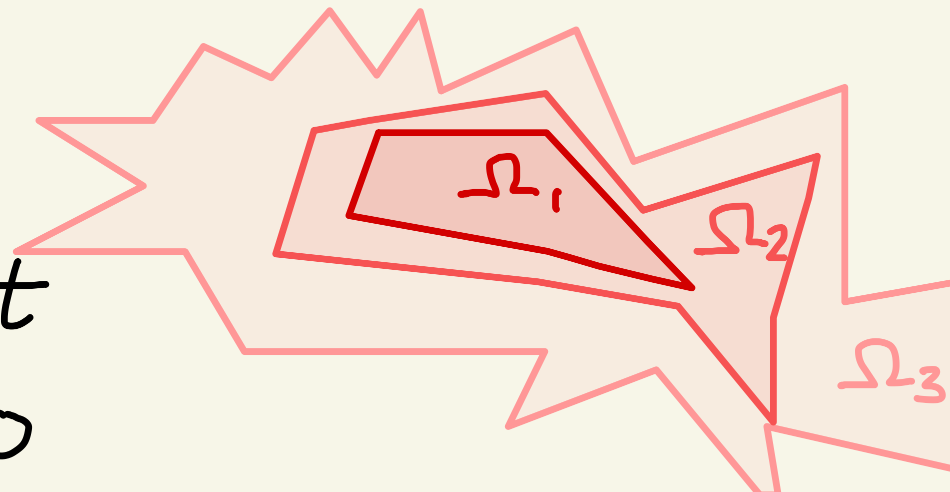

Then
$V_j \xrightarrow{M} V := \overline{\bigcup_j V_j} = \overline{\bigcup_j \widetilde{H}^s(\Omega_j)}$

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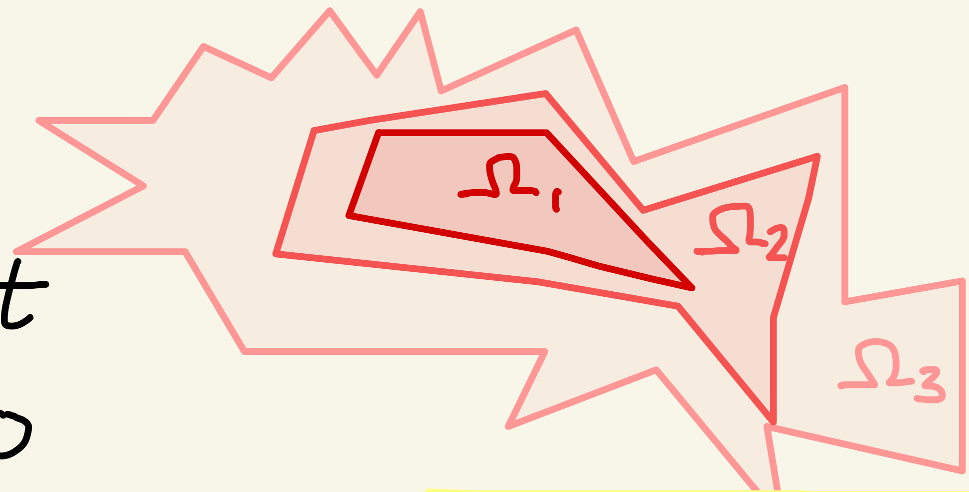


Then

$V_j \xrightarrow{M} V := \overline{\bigcup_j V_j} = \overline{\bigcup_j \widetilde{H}^s(\Omega_j)}$

Ex show $V = \widetilde{H}^s\left(\bigcup_j \Omega_j\right)$

Sequences of subspaces and Mosco convergence

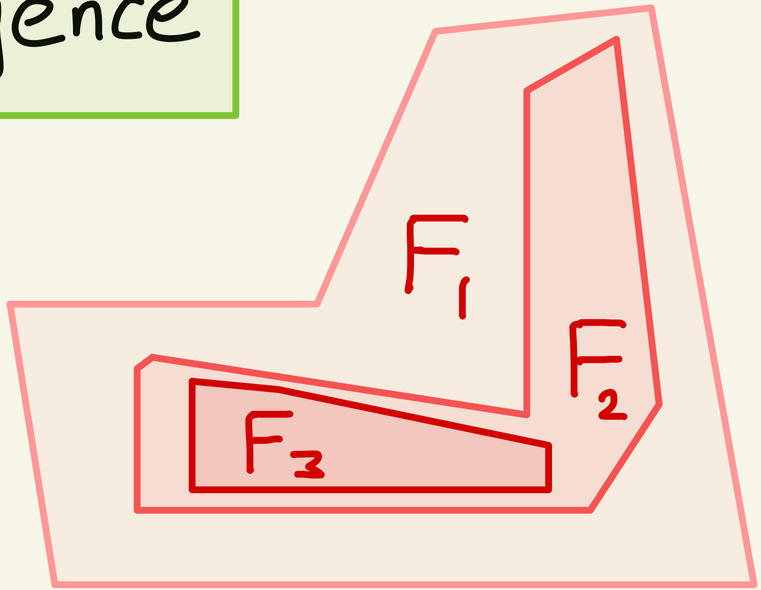
Suppose

$F_1 \supset F_2 \supset$

are closed and let

$V_j := H^s_{F_j}$ so

$V_1 \supset V_2 \supset$



Then

$$V_j \xrightarrow{M} V := \bigcap_j V_j = \bigcap_j H^s_{F_j} = H^s_F, \quad F := \bigcap_j F_j$$

Sequences of subspaces and Mosco convergence

Suppose

$\Omega_1 \supset \Omega_2 \supset$

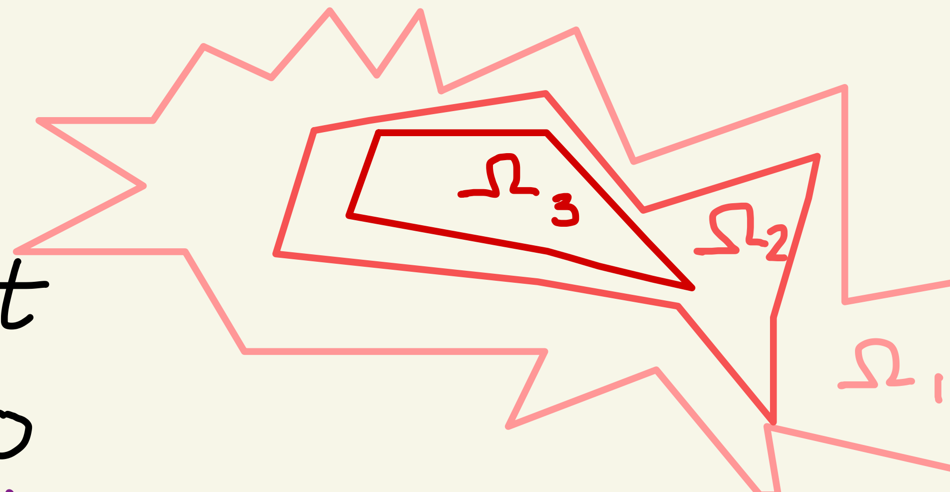
are open and let

$V_j := \widetilde{H}^s(\Omega_j)$ so

$V_1 \supset V_2 \supset$

Then

$V_j \xrightarrow{M} V := \bigcap_j V_j = \bigcap_j \widetilde{H}^s(\Omega_j) = ??$



Part 4: Integral Equation Formulations for Scattering by Thin Screens*

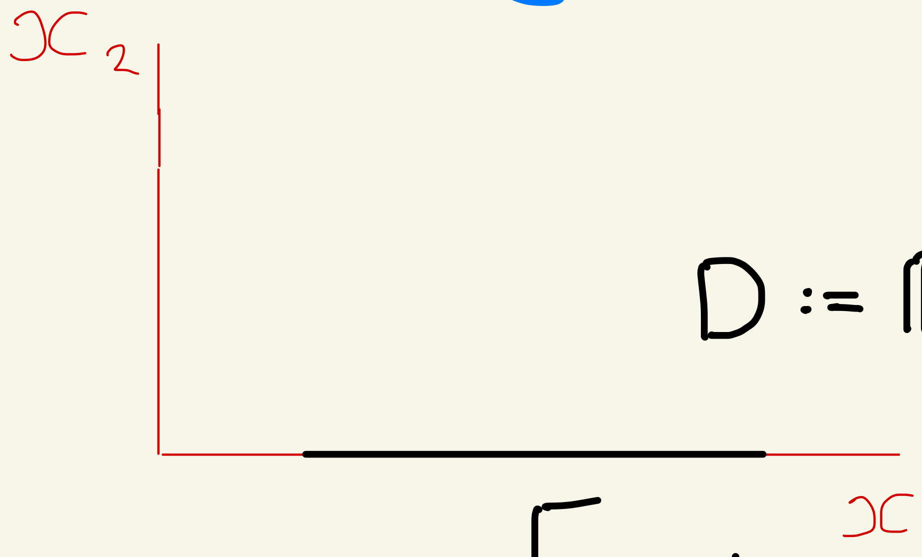
* which may be fractal!

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,

find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$

st

$$\Delta u + k^2 u = g \text{ in } D$$



$k \in \mathbb{C}$,

$\operatorname{Im} k > 0$

$D := \mathbb{R}^2 \setminus \Gamma$

Γ, closed

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st

$$\Delta u + k^2 u = g \text{ in } D$$

$\Leftrightarrow$ Find $u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle$, $\forall v \in \tilde{H}^1(D)$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,

find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$

st

$$\Delta u + k^2 u = g \text{ in } D$$

$\Leftrightarrow$ Find $u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle$, $\forall v \in \tilde{H}^1(D)$

Applying L-M these have exactly one soln and $\|u\| \leq c(k) \, \|g\|$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,

find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$

st

$$\Delta u + k^2 u = g \text{ in } D$$

$$\iff \boxed{\text{Find } u \in \tilde{H}^1(D) \text{ st } a(u,v) = \langle g, \bar{v} \rangle,\ \forall v \in \tilde{H}^1(D)}$$

Applying L-M these have exactly one soln

and $\|u\| \leq c(k)\, \|g\|$ TRUE IF $D = \mathbb{R}^2$!

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in H^1(\mathbb{R}^2)$
st

$$\Delta u + k^2 u = g \text{ in } \mathbb{R}^2$$

$\iff$ $\boxed{\text{Find } u \in H^1(\mathbb{R}^2) \text{ st } a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in H^1(\mathbb{R}^2)}$

Applying L-M these have exactly one soln
and $\|u\| \leq c(k) \|g\|$ TRUE IF $D = \mathbb{R}^2$!

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in H^1(\mathbb{R}^2)$
st
$$\Delta u + k^2 u = g \text{ in } \mathbb{R}^2$$

$\Longleftrightarrow$ Find $u \in H^1(\mathbb{R}^2)$ st $a(u,v) = \langle g, \bar{v} \rangle$, $\forall v \in H^1(\mathbb{R}^2)$

Applying L-M these have exactly one soln and $\|u\| \leq c(k)\,\|g\|$ So $\Delta + k^2 \; H^1(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2)$ is isomorphism with inverse $G = (\Delta + k^2)^{-1}$ & $\|G\| \leq c(k)$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in H^1(\mathbb{R}^2)$
st

$$\Delta u + k^2 u = g \iff u = Gg$$

$$\iff \boxed{\text{Find } u \in H^1(\mathbb{R}^2) \text{ st } a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in H^1(\mathbb{R}^2)}$$

Applying L-M these have exactly one soln and $\|u\| \leqslant c(k)\,\|g\|$. So $\Delta + k^2 : H^1(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2)$ is isomorphism with inverse $G = (\Delta + k^2)^{-1}$ & $\|G\| \leqslant c(k)$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$
st
$$\Delta u + k^2 u = g \text{ in } D$$

Let $\phi = \Delta u + k^2 u - g$. Then $\phi = 0$ in D, so
$\phi \in H^{-1}_\Gamma$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
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st
$$\Delta u + k^2 u = g \text{ in } D$$

Let $\phi = \Delta u + k^2 u - g$. Then $\phi = 0$ in D, so $\phi \in H^{-1}_\Gamma$, and

$$G\phi = G(\Delta + k^2)u - Gg = u - Gg$$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$
st
$$\Delta u + k^2 u = g \text{ in } D$$

Let $\phi = \Delta u + k^2 u - g$. Then $\phi = 0$ in D, so $\phi \in H_\Gamma^{-1}$, and
$$G\phi = G(\Delta + k^2)u - Gg = uG \quad g$$

Now $\langle \psi, v \rangle = 0$, $\forall v \in C_0^\infty(D)$, $\psi \in H_\Gamma^{-1}$

$$\boxed{\begin{array}{l}\text{Given } g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*, \\ \text{find } u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)} \\ \text{st} \qquad \Delta u + k^2 u = g \text{ in } D\end{array}}$$

Let $\phi = \Delta u + k^2 u - g$ Then $\phi = 0$ in D, so $\phi \in H^{-1}_{\Gamma}$, and

$$G\phi = G(\Delta + k^2)u - Gg = u - Gg$$

Now $\langle \psi, v \rangle = 0$, $\forall v \in \cancel{C_0^\infty}(D)$, $\psi \in H^{-1}_{\Gamma}$

$\tilde{H}^1(D)$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$
st
$$\Delta u + k^2 u = g \text{ in } D$$

Let $\phi = \Delta u + k^2 u - g$. Then $\phi = 0$ in D, so $\phi \in H^{-1}_\Gamma$, and
$$G\phi = G(\Delta + k^2)u - Gg = u - Gg$$

Now $\langle \psi, v \rangle = 0$, $\forall v \in \cancel{C_0^\infty(D)}$ $\tilde{H}^1(D)$, $\psi \in H^{-1}_\Gamma$

So $\langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle$,
$$\forall \psi \in H^{-1}_\Gamma$$

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Let $\phi = \Delta u + k^2 u - g$. Then $\phi = 0$ in D, so $\phi \in H^{-1}_\Gamma$, and

$$G\phi = G(\Delta + k^2)u - Gg = u - Gg$$

Now $\langle \psi, v \rangle = 0$, $\forall v \in \tilde{H}^1(D)$, $\psi \in H^{-1}_\Gamma$

$$\boxed{\text{Find } \phi \in H^{-1}_\Gamma \text{ st } \langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \ \forall \psi \in H^{-1}_\Gamma}$$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$
st
$$\Delta u + k^2 u = g \text{ in } D$$

Let $\phi = \Delta u + k^2 u - g$ Then $\phi = 0$ in D, so $\phi \in H^{-1}_\Gamma$, and

$$G\phi = G(\Delta + k^2)u - Gg = u - Gg$$

Now $\langle \psi, v \rangle = 0, \ \forall v \in \tilde{H}^1(D), \psi \in H^{-1}_\Gamma$

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Our INTEGRAL EQ VP

(1) $\boxed{\begin{array}{l}\text{Given } g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*, \\ \text{find } u \in \tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)} \\ \text{st} \\ \qquad \Delta u + k^2 u = g \text{ in } D\end{array}}$

(2) $\boxed{\text{Find } \phi \in H^{-1}_\Gamma \text{ st } \langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \ \forall \psi \in H^{-1}_\Gamma}$

Thm If u satisfies (1) then

$\phi = \Delta u + k^2 u - g$ satisfies (2) and

$$u = Gg + G\phi$$

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(2) $\boxed{\text{Find } \phi \in H_\Gamma^{-1} \text{ st } \langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \ \forall \psi \in H_\Gamma^{-1}}$

Thm If u satisfies (1) then

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Conversely, if ϕ satisfies (2) then u given by $(+)$ satisfies (1)

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(2) $\boxed{\text{Find } \phi \in H^{-1}_\Gamma \text{ st } \langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \ \forall \psi \in H^{-1}_\Gamma}$

Thm If u satisfies (1) then $\phi = \Delta u + k^2 u - g$ satisfies (2) and $u = Gg + G\phi$ (+) } (A) proved already

Conversely, if ϕ satisfies (2) then u given by (+) satisfies (1) } Follows from (A) and well-posedness of (2)

Well-posedness of our IE formulation

Given $g \in H^{-1}(\mathbb{R}^2)$ find $\phi \in H^{-1}_{\Gamma}$ s t $\langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \ \forall \psi \in H^{-1}_{\Gamma}$

$\Longleftrightarrow$ Find $\phi \in H^{-1}_{\Gamma}$ s t $A(\phi, \psi) = -\langle \bar{\psi}, Gg \rangle, \ \forall \psi \in H^{-1}_{\Gamma}$

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Continuous? Coercive?

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Continuous?

$\forall\, \phi,\psi \in H^{-1}_{\Gamma}$,

$$|A(\phi,\psi)| = |\langle \bar{\psi}, G\phi \rangle| \leq \|\psi\|\ \|G\phi\|$$

$$\leq c(k)\ \|\psi\|\ \|\phi\|$$

$$A(\phi,\psi) = \langle \bar{\psi}, G\phi \rangle \qquad \phi,\psi \in H^{-1}_{\Gamma}$$

Coercive?

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
find $u \in H^1(\mathbb{R}^2)$
st

$$\Delta u + k^2 u = g \Leftrightarrow u = Gg$$

$\Leftrightarrow$ Find $u \in H^1(\mathbb{R}^2)$ st $a(u,v) = \langle g, v\rangle$, $\forall v \in H^1(\mathbb{R}^2)$

$$a(u,v) = \int_{\mathbb{R}^2} \left(k^2 u \bar{v} - \nabla u \, \nabla \bar{v}\right)$$

Given $g \in H^{-1}(\mathbb{R}^2) := (H^1(\mathbb{R}^2))^*$,
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$$\Delta u + k^2 u = g \iff u = Gg$$

$\iff$ Find $u \in H^1(\mathbb{R}^2)$ st $a(u,v) = \langle g, \bar{v} \rangle$, $\forall v \in H^1(\mathbb{R}^2)$

coercive

Soln is $u = Gg$ so

$$a(Gg, Gg) = \langle g, \overline{Gg} \rangle$$

$A(\phi,\psi) = \langle \bar{\psi}, G\phi \rangle \quad \phi,\psi \in H^{-1}_{\Gamma}$

Coercive?

$\forall\, \phi \in H^{-1}_{\Gamma}$,

$$|A(\phi,\phi)| = |\langle \bar{\phi}, G\phi \rangle| \\ = |a(G\phi, G\phi)|$$

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Coercive?

$\forall\, \phi \in H^{-1}_{\Gamma}$,

$$|A(\phi,\phi)| = |\langle \bar{\phi}, G\phi \rangle|$$

$$= |a(G\phi, G\phi)|$$

$$\geq c(k)\, \|G\phi\|^2$$

Coercivity of $a(\,,\,)$

$A(\phi,\psi) = \langle \bar{\psi}, G\phi \rangle \quad \phi,\psi \in H^{-1}_{\Gamma}$

Coercive?

Since

$\|\phi\| = \|(\Delta+k^2)G\phi\|$

$\leq \|\Delta+k^2\| \, \|G\phi\|$

$\forall \phi \in H^{-1}_{\Gamma}$,

$$|A(\phi,\phi)| = |\langle \bar{\phi}, G\phi \rangle|$$

$$= |a(G\phi, G\phi)|$$

$$\geq c(k) \|G\phi\|^2 \geq \tilde{c}(k) \|\phi\|^2$$

Well-posedness of our IE formulation

Given $g \in H^{-1}(\mathbb{R}^2)$ find $\phi \in H^{-1}_\Gamma$ s t $\langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle$, $\forall \psi \in H^{-1}_\Gamma$

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Cont & coercive so

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Thm $\exists$ exactly one soln $\phi \in H^{-1}_{\Gamma}$ and

$$\|\phi\| \leqslant c(k) \|Gg\| \leqslant \tilde{c}(k) \|g\|$$

Proof L-M

Well-posedness of our IE formulation

Given $g \in H^{-1}(\mathbb{R}^2)$ find $\phi \in H^{-1}_{\Gamma}$ s.t. $\langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle,\ \forall \psi \in H^{-1}_{\Gamma}$

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OK. but where is the boundary integral?

$\boxed{\text{Thm}}$ (eg McLean, 2000, p104)

$$H^{-1}_{\Gamma}(\mathbb{R}^2) = \{\gamma^*\phi \cdot \phi \in H^{-1/2}_{\Gamma}(\mathbb{R})\}$$

where $\gamma^* \; H^{-1/2}(\mathbb{R}) \to H^{-1}(\mathbb{R}^2)$,

defined by

$$\langle \gamma^*\phi, \psi \rangle := \langle \phi, \gamma\psi \rangle, \quad \phi \in H^{-1/2}(\mathbb{R}), \psi \in H^{1}(\mathbb{R}^2)$$

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where $\gamma : H^1(\mathbb{R}^2) \to H^{1/2}(\mathbb{R})$ is trace op,

$$\gamma\phi(x_1) := \phi((x_1, 0)), \quad x_1 \in \mathbb{R}, \phi \in C_0^\infty(\mathbb{R}^2)$$

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$$\langle \gamma^* \tilde{\psi}, G \gamma^* \tilde{\phi} \rangle = \langle \gamma^* \tilde{\psi}, Gg \rangle, \quad \tilde{\psi} \in H^{-1/2}_{\Gamma}(\mathbb{R})$$

Given $g \in H^{-1}(\mathbb{R}^2)$ find $\phi \in H_\Gamma^{-1}$ st $\langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \forall \psi \in H_\Gamma^{-1}$

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$$\boxed{\text{Given } g \in \tilde{H}^{-1}(\mathbb{R}^2) \text{ find } \phi \in \tilde{H}^{-1}_{\Gamma} \text{ s.t. } \langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle, \ \forall \psi \in H^{-1}_{\Gamma}}$$

$\Leftrightarrow$ Find $\tilde{\phi} \in H^{-1/2}_{\Gamma}(\mathbb{R})$ s.t.

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where

$$S = \gamma G \gamma^* \quad H^{-1/2}(\mathbb{R}) \to H^{1/2}(\mathbb{R})$$

is the (acoustic) single-layer potential operator on $\Gamma_\infty = \{(x_1, 0) \ x_1 \in \mathbb{R}\} \simeq \mathbb{R}$

Given $g \in H^{-1}(\mathbb{R}^2)$ find $\phi \in H^{-1}_{\Gamma}$ s t $\langle \psi, G\phi \rangle = -\langle \psi, Gg \rangle$, $\forall \psi \in H^{-1}_{\Gamma}$

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where $S = \gamma G \gamma^* \quad H^{-1/2}(\mathbb{R}) \to H^{1/2}(\mathbb{R})$

OK, but still, where are the INTEGRALS !!

Remember

$$u = Gg$$

$$\Longleftrightarrow \Delta u + k^2 u = g$$

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$$\Longleftrightarrow (k^2 - \xi^2)\hat{u}(\xi) = \hat{g}(\xi)$$

$$\Longleftrightarrow \quad u = \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{k^2 - \xi^2}\right)$$

Remember $u = Gg$

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$$\Leftrightarrow \quad u = \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{k^2 - \xi^2}\right)$$

$$\Rightarrow u(x) = \langle g, \Phi(x - \cdot) \rangle$$

if $x \notin \operatorname{supp}(g)$

or $g \in H^{-1+\epsilon}(\mathbb{R}^2)$

Hankel fn

where $\Phi(x) = -\frac{1}{4} \boxed{H_0^{(1)}}(k|x|) \in L^1(\mathbb{R}^2)$

Remember $u = Gg$

$$\Leftrightarrow \quad u = \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{k^2 - \xi^2}\right)$$

$$\Rightarrow u(x) = \langle g, \Phi(x - \cdot)\rangle$$

if $x \notin \mathrm{supp}(g)$

or $g \in H^{-1+\epsilon}(\mathbb{R}^2)$

$$\underset{(g \in L^2(\mathbb{R}^2))}{=} \int_{\mathbb{R}^2} g(y)\Phi(x-y)\,dy$$

Thus, if $\tilde{\phi} \in L^2(\mathbb{R})$,

$$S\tilde{\phi}(x_1) = (\gamma G \gamma^* \tilde{\phi})(x_1)$$
$$= \int_{\mathbb{R}} \tilde{\Phi}(x_1 - y_1)\, \tilde{\phi}(y_1)\, dy_1$$

where $$\tilde{\Phi}(t) = -\frac{1}{4} H_0^{(1)}(k|t|), \quad t \in \mathbb{R}$$