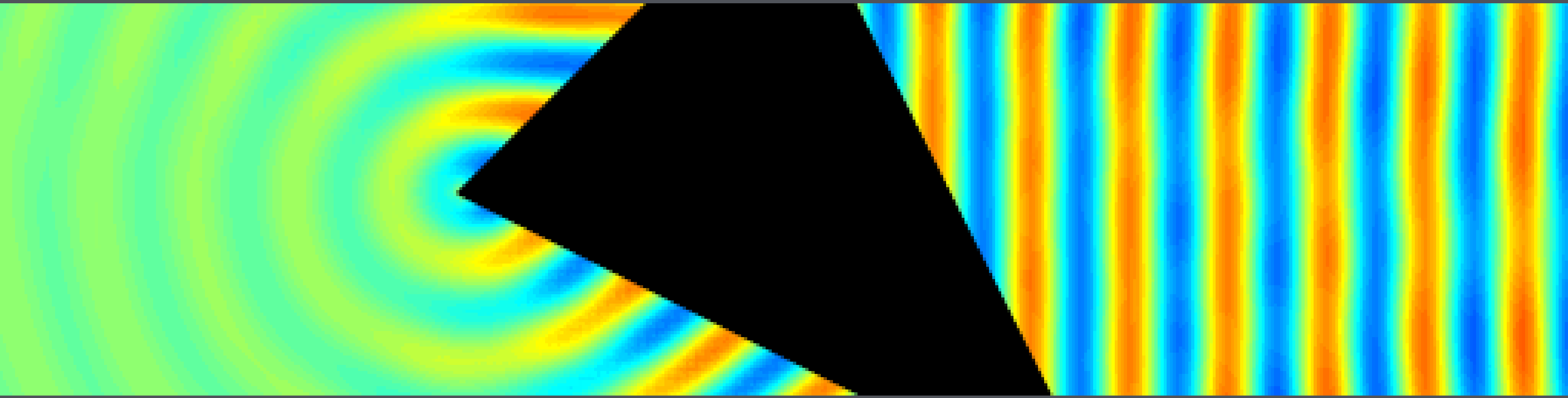
University of Reading

INTRODUCTION TO THE BOUNDARY ELEMENT METHOD IN ACOUSTICS: UK ACOUSTICS NETWORK WEBINAR, MAY 2020



Simon Chandler-Wilde

Professor of Applied Maths

www.reading.ac.uk/~sms03snc

ABOUT ME

- My PhD and background in **outdoor noise propagation** and **noise barriers**, working with **David Hothersall, Kirill Horoshenkov** at Bradford.
- 1996 **Tyndall Medal** of the Institute of Acoustics.
- Currently Professor of Applied Mathematics at Reading, working particularly on **Numerical/Asymptotic Boundary Element Methods in Acoustics**, with collaborators/postdocs including **Steve Langdon** (Brunel), **Dave Hewett**, **Timo Betcke** (UCL), **Andrew Moiola** (Pavia).

WHAT WILL I TALK ABOUT?

1. The Wave Equation, and its time harmonic version, the Helmholtz equation
2. Fundamental solutions
3. A first BEM example: propagation through an aperture
4. General 2D and 3D BEM
5. When is BEM a good method to use?
6. Further reading

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The Wave Equation and Helmholtz Equation

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

If time-dependence is **time harmonic**, i.e., where $\mathbf{r} = (x, y, z)$,

$$U(\mathbf{r}, t) = A(\mathbf{r})\cos(\phi(\mathbf{r}) - \omega t),$$

for some $\omega = 2\pi f > 0$, with $f =$ **frequency**, then

$$U(\mathbf{r}, t) = \Re\left(u(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

where $u(\mathbf{r}) = A(\mathbf{r})\exp(\mathrm{i}\phi(\mathbf{r}))$ satisfies the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

with $k = \omega/c$ the **wavenumber**.

If time-dependence is **time harmonic** then

$$U(\mathbf{r},t) = \Re\left(u(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

for some $\omega = 2\pi f > 0$, with $f =$ **frequency**, where u satisfies the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

with $k = \omega/c$ the **wavenumber**. E.g. if

$$u(\mathbf{r}) = \mathrm{e}^{\mathrm{i}k\mathbf{r}\cdot\mathbf{d}},$$

for some **unit vector** $\mathbf{d}$, then

$$U(\mathbf{r},t) = \Re\left(u(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right) = \cos(k\mathbf{r}\cdot\mathbf{d} - \omega t)$$

is a **plane wave** travelling in direction $\mathbf{d}$ with **wavelength**

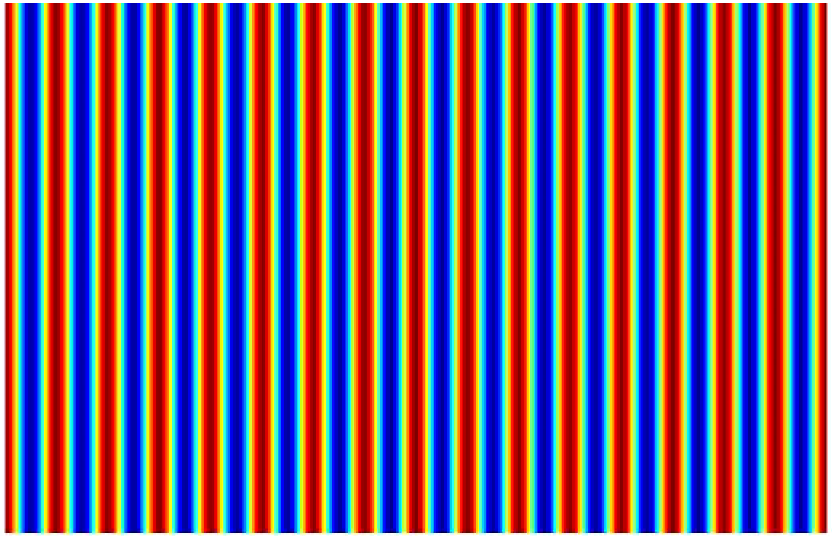
$$\lambda = \frac{2\pi}{k} = \frac{c}{f}.$$

E.g. when $\mathbf{d} = (1, 0, 0)$ then

$$u(\mathbf{r}) = \mathrm{e}^{\mathrm{i}k\mathbf{r}\cdot\mathbf{d}} = \mathrm{e}^{\mathrm{i}kx} \quad \text{and} \quad U(\mathbf{r}, t) = \Re\left(u(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right) = \cos(kx - \omega t),$$

a **plane wave** travelling in the x direction with **wavelength**

$$\lambda = \frac{2\pi}{k} = \frac{c}{f}.$$



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Fundamental Solution of the Helmholtz Equation (in 2D)

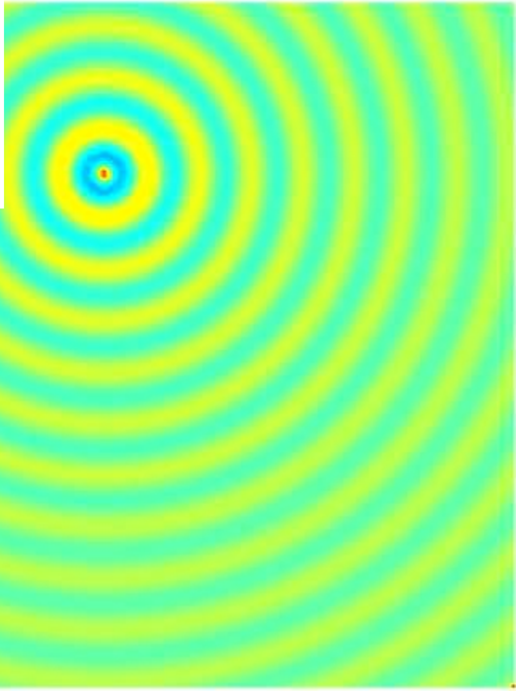
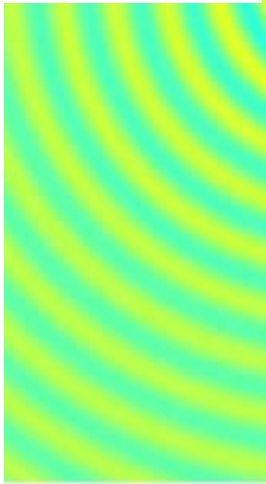
When the acoustic pressure is generated by a **line source** along the z-axis, the solution to the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

depends only on x and y. At $\mathbf{r} = (x, y)$ the solution is

$$u(\mathbf{r}) = \frac{\mathrm{i}}{4} H_0^{(1)}(kr) \approx \mathrm{const.} \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}, \quad \text{where } r = \sqrt{x^2 + y^2},$$

and $H_0^{(1)}$ is the **Hankel function of the first kind of order zero** (a Bessel function).

The corresponding solution of the wave equation is

$$\begin{aligned} U(\mathbf{r}, t) &= \Re\left(u(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right) \\ &= \Re\left(\frac{\mathrm{i}}{4} H_0^{(1)}(kr)\mathrm{e}^{-\mathrm{i}\omega t}\right) \\ &\approx \mathrm{const.} \frac{\cos(kr - \omega t)}{\sqrt{kr}}. \end{aligned}$$

Fundamental Solution of the Helmholtz Equation (in 2D)

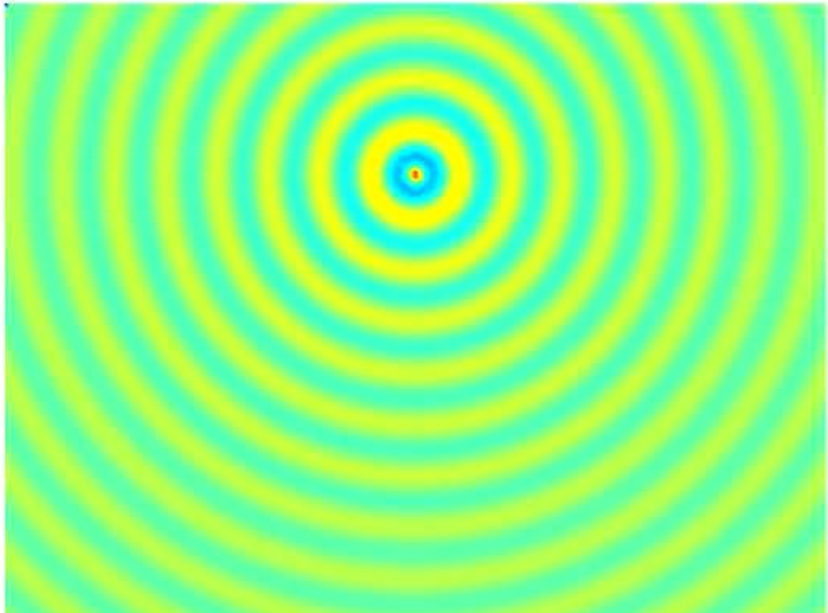
If the **line source** is parallel to the z-axis through $\mathbf{r}_0 = (x_0, y_0)$, the solution to the Helmholtz equation is

$$u(\mathbf{r}) = \Phi(\mathbf{r}, \mathbf{r}_0) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_0|) = \frac{\mathrm{i}}{4} H_0^{(1)}(kR)$$

where

$$R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

The function $\Phi(\mathbf{r}, \mathbf{r}_0)$, which depends on where we are measuring $(\mathbf{r})$ and where the source is $(\mathbf{r}_0)$, is called a **fundamental solution of the Helmholtz equation**.



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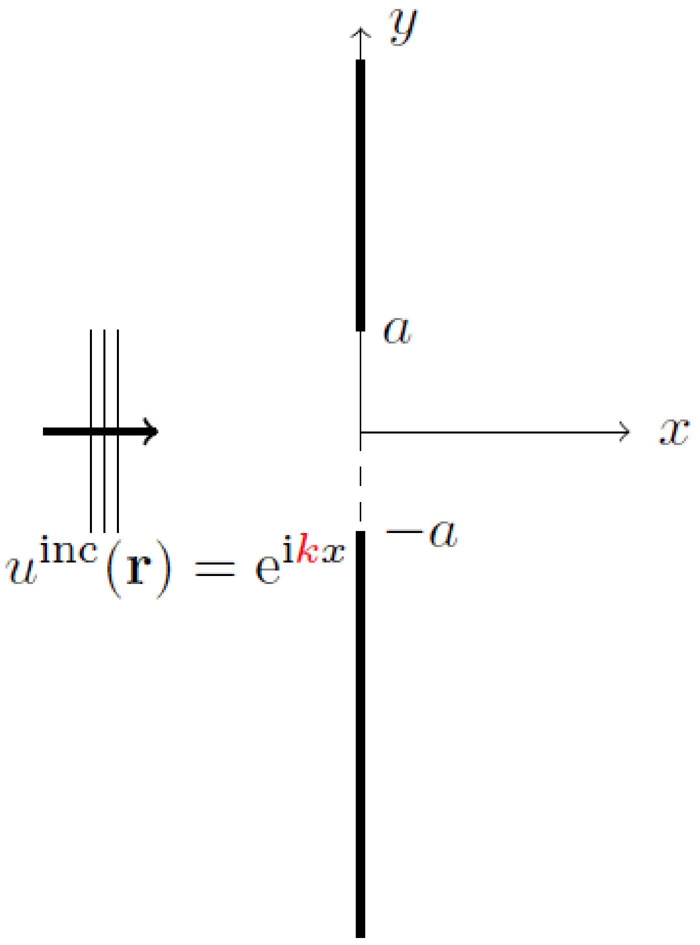
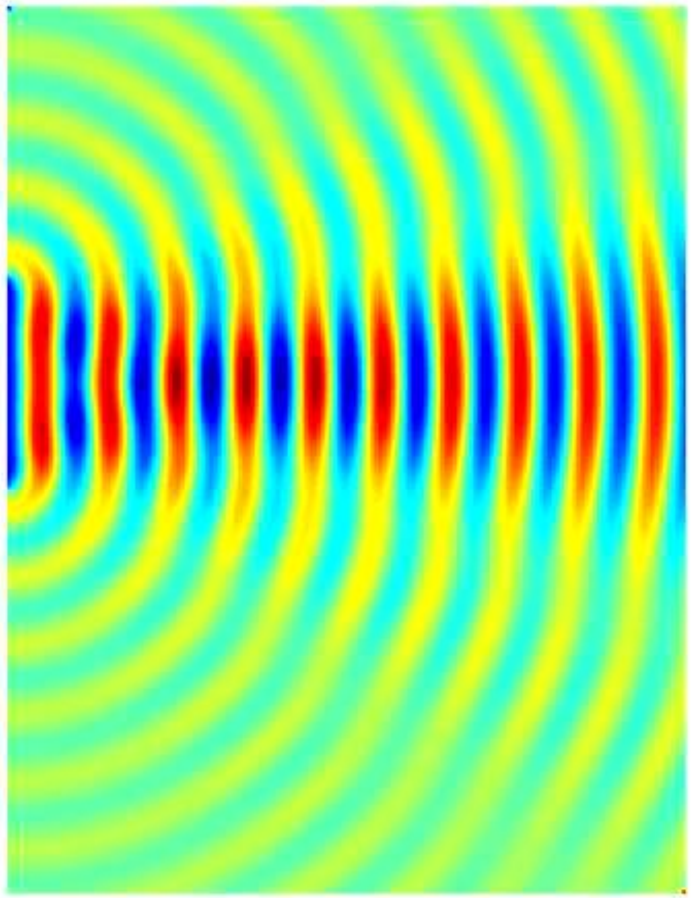
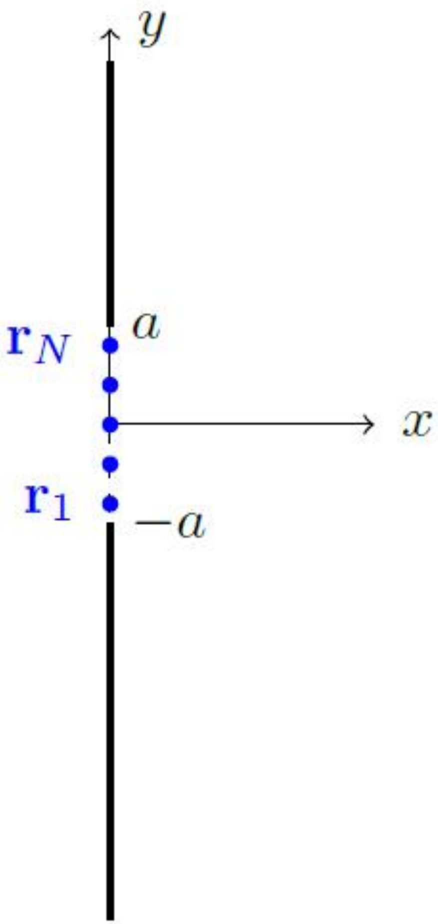
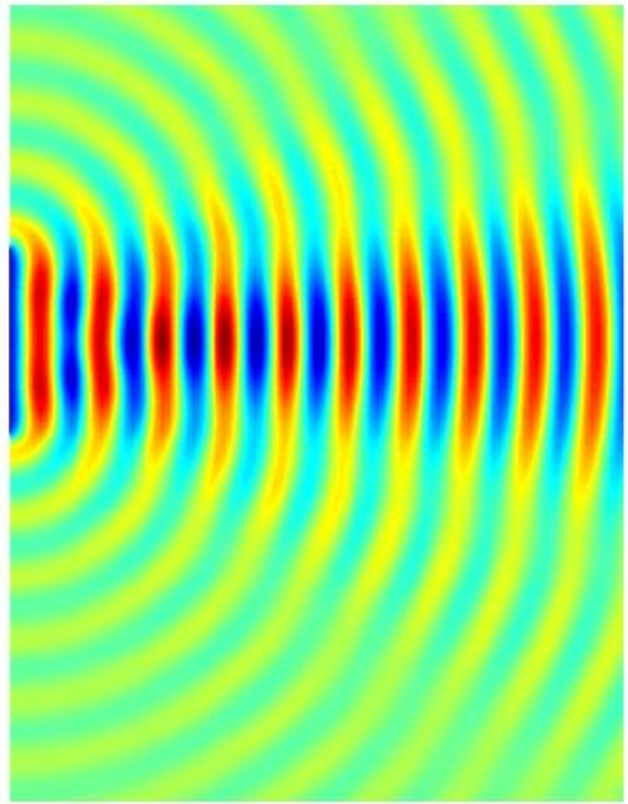
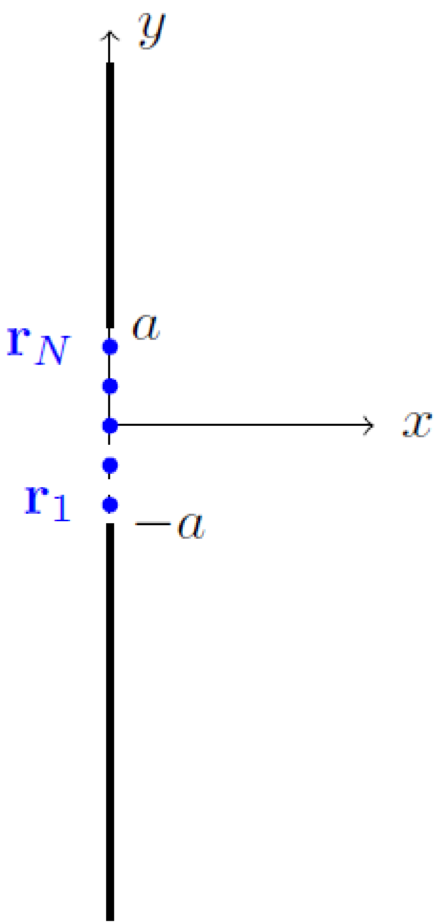
A First BEM Example: Diffraction Through an Aperture in a Rigid Screen






y
a
$\mathbf{r}_N$
x
$\mathbf{r}_1$
$-a$





Approximate u to the right of the aperture by

$$u(\mathbf{r}) = \sum_{n=1}^{N} a_n \Phi(\mathbf{r}, \mathbf{r}_n),$$

where the points $\mathbf{r}_1, ..., \mathbf{r}_N$ are equally spaced in the aperture, distance $h = 2a/N$ apart, precisely

$$\mathbf{r}_n = (0, y_n) \quad \text{with} \quad y_n = -a + (n - 0.5)h.$$

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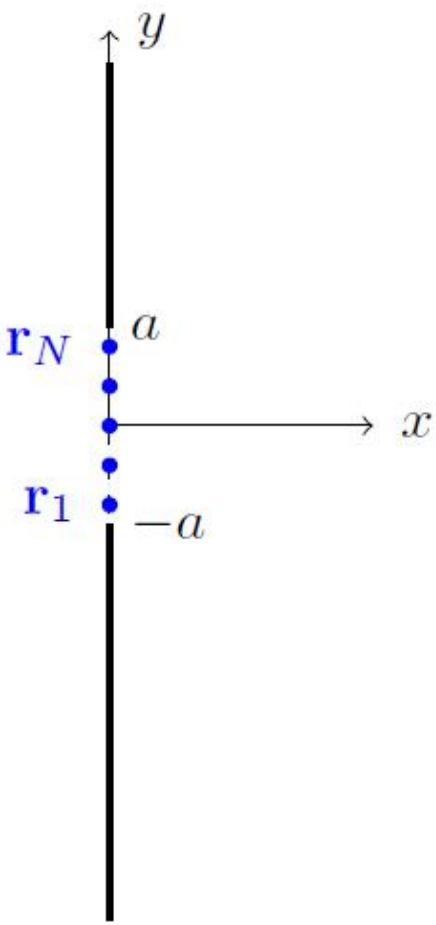

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$$\mathbf{r}_n = (0, y_n) \quad \text{with} \quad y_n = -a + (n - 0.5)h.$$

If we take each $a_n = h\phi(y_n)$, for some continuous function ϕ, and let $N \to \infty$, we get

$$u(\mathbf{r}) = \int_{-a}^{a} \Phi(\mathbf{r}, (0, y_s))\phi(y_s)\,\mathrm{d}y_s,$$

a continuous distribution of sources in the aperture with **density** ϕ.

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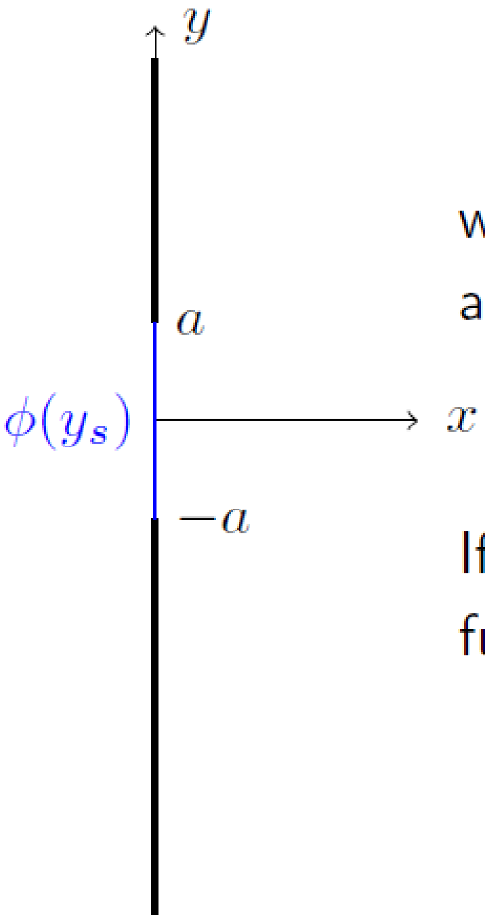

where the points $\mathbf{r}_1, \ldots, \mathbf{r}_N$ are equally spaced in the aperture, distance $h = 2a/N$ apart, precisely

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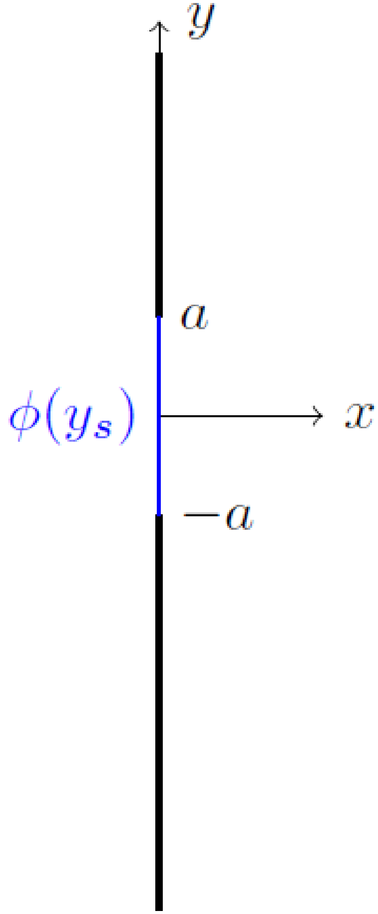
To the right of the aperture



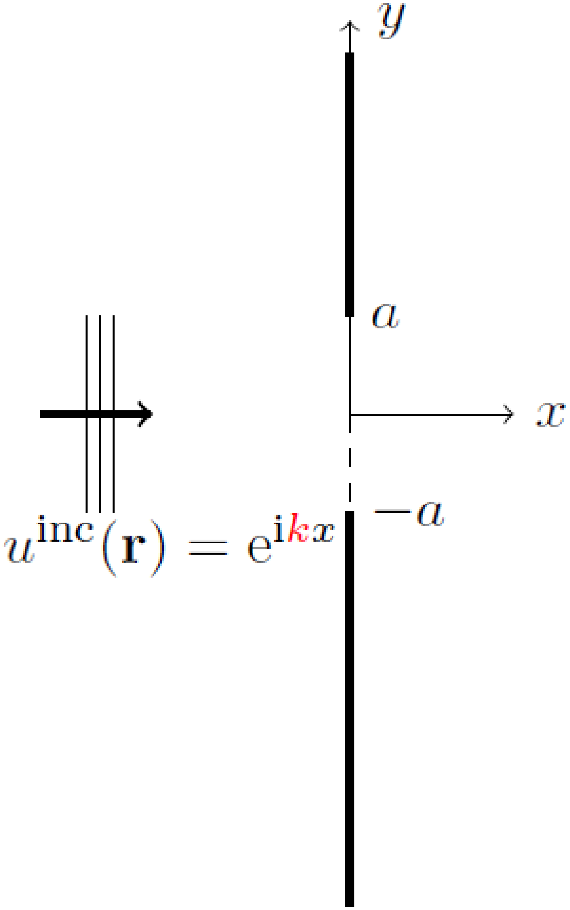
$$
\begin{aligned}
u(\mathbf{r}) &= \int_{-a}^{a} \Phi(\mathbf{r}, (0, y_s))\phi(y_s)\,\mathrm{d}y_s \\
&= \frac{\mathrm{i}}{4} \int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2 + (y - y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s,
\end{aligned}
$$

for some **source density** ϕ. This satisfies the Helmholtz/wave equation and has the correct radiating behaviour at infinity. Also, for every $\mathbf{r} = (x, y)$ not in the actual aperture,

$$
\frac{\partial u(\mathbf{r})}{\partial x} = -\frac{kx\mathrm{i}}{4} \int_{-a}^{a} \frac{H_1^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)}{\sqrt{x^2+(y-y_s)^2}}\phi(y_s)\,\mathrm{d}y_s,
$$

which is zero when $x = 0$ and $|y| > a$, as it should be for a rigid screen. **But can we also choose ϕ to give the correct field in the aperture?**

What is the field in an aperture in a rigid screen?



Let's add another incident field $\mathrm{e}^{-\mathrm{i}kx}$.

(i) By symmetry, this will double the field in the aperture itself;

(ii) The complete solution will be simply

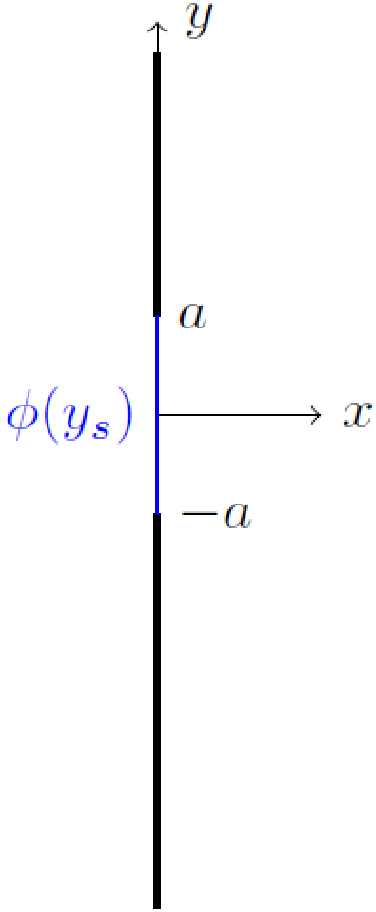
$$u(\mathbf{r}) = \mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx};$$

this satisfies rigid no-flow condition ($\partial u/\partial x = 0$) on the screen.

(iii) So the field in the aperture for the original incident field is

$$\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k0} + \mathrm{e}^{-\mathrm{i}k0}\right) = 1.$$

A first Boundary Integral Equation



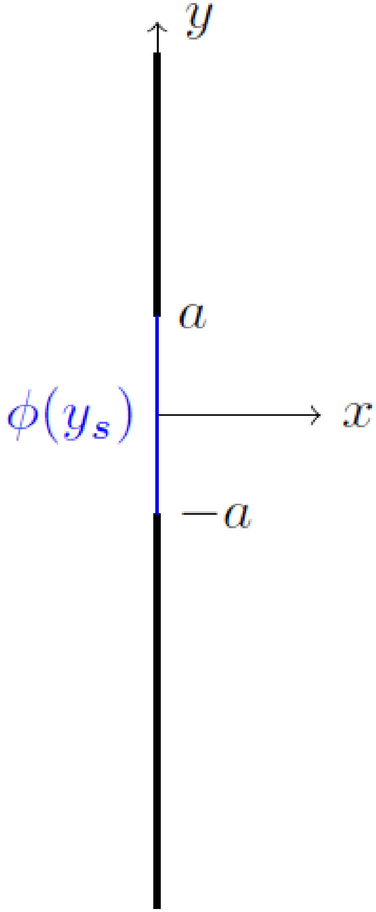
$$
\begin{aligned}
u(\mathbf{r}) &= \int_{-a}^{a} \Phi(\mathbf{r}, (0, y_s))\phi(y_s)\,\mathrm{d}y_s \\
&= \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2 + (y-y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s.
\end{aligned}
$$

This gives $u(\mathbf{r}) = 1$ in the aperture provided

$$
1 = \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k|y-y_s|\right)\phi(y_s)\,\mathrm{d}y_s,
$$

for $-a < y < a$, **an integral equation** to determine ϕ.

A first Boundary Integral Equation



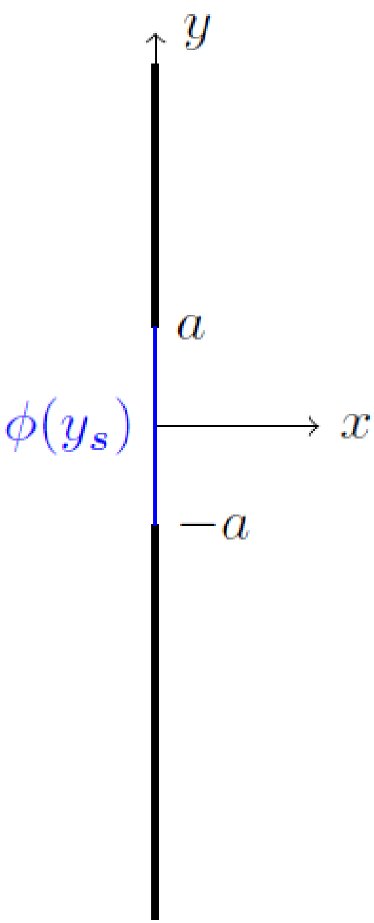
$$u(\mathbf{r}) = \int_{-a}^{a} \Phi(\mathbf{r}, (0, y_s))\phi(y_s)\,\mathrm{d}y_s$$
$$= \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2 + (y - y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s.$$

This gives $u(\mathbf{r}) = 1$ in the aperture provided

$$1 = \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k|y - y_s|\right)\phi(y_s)\,\mathrm{d}y_s,$$

for $-a < y < a$, **an integral equation** to determine ϕ. We proceed by: i) solving this integral equation to determine ϕ; ii) using ϕ in the top equations to determine $u(\mathbf{r})$ away from the aperture.

What is the meaning of ϕ?



$$u(\mathbf{r}) = \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s.$$

so, for $x > 0$,

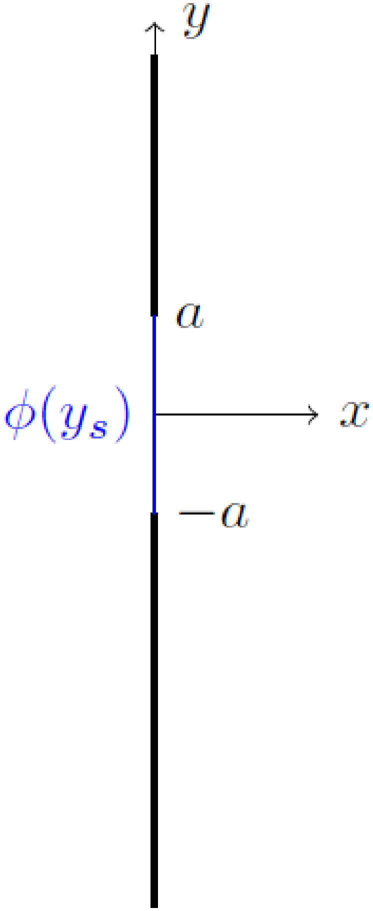
$$\frac{\partial u(\mathbf{r})}{\partial x} = -\frac{kx\mathrm{i}}{4}\int_{-a}^{a}\frac{H_1^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)}{\sqrt{x^2+(y-y_s)^2}}\phi(y_s)\,\mathrm{d}y_s.$$

Thus, as $x \to 0$ with $-a < y < a$ – this a really nice maths exercise in limits –

$$\frac{\partial u(\mathbf{r})}{\partial x} \to -\frac{1}{2}\phi(y),$$

so $\phi = -2\frac{\partial u}{\partial x}$ on the aperture.

The (high frequency) Kirchhoff approximation



$$
\begin{aligned}
u(\mathbf{r}) &= \int_{-a}^{a} \Phi(\mathbf{r}, (0, y_s))\phi(y_s)\,\mathrm{d}y_s \\
&= \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s .
\end{aligned}
$$

with

$$
\phi = -2\frac{\partial u}{\partial x} \approx -2\frac{\partial u^{\mathrm{inc}}}{\partial x} = -2\mathrm{i}k,
$$

so that

$$
\begin{aligned}
u(\mathbf{r}) &\approx -2\int_{-a}^{a} \Phi(\mathbf{r}, (0, y_s))\frac{\partial u^{\mathrm{inc}}}{\partial x}(0, y_s)\,\mathrm{d}y_s \\
&= \frac{k}{2}\int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)\mathrm{d}y_s ,
\end{aligned}
$$

this the **Kirchhoff approximation**.

Lord Rayleigh, "Theory of Sound", 2nd Ed., Vol. II, Macmillan, New York, 1896: the 19th century mathematics of screens and apertures, pp.139-140.

If $P\cos(nt+\epsilon)$ denote the value of $d\phi/dx$ at the various points of the area (S) of the aperture, the condition for determining P and ϵ is by (6) § 278,

$$-\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS=\cos nt \quad\text{.........(2)},$$

where r denotes the distance between the element dS and any fixed point in the aperture. When P and ϵ are known, the complete value of ϕ for any point on the positive side of the screen is given by

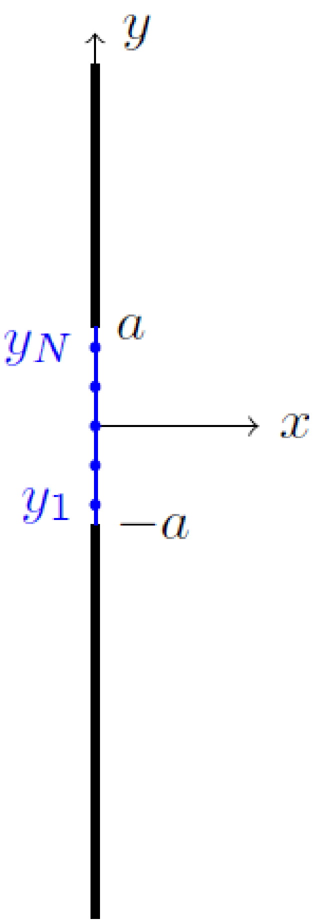
$$\phi=-\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS \quad\text{..........................(3)},$$

and for any point on the negative side by

$$\phi=+\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS+2\cos nt\cos kx \quad\text{...... (4)}.$$

The expression of P and ϵ for a finite aperture, even if of circular form, is probably beyond the power of known methods; but in the

A first Boundary Element Method



For $-a < y < a$, where $h = (2a)/N$, $y_n = -a + (n - 0.5)h$,

$$
\begin{aligned}
1 &= \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k|y-y_s|\right)\phi(y_s)\,\mathrm{d}y_s \\
&= \sum_{n=1}^{N}\frac{\mathrm{i}}{4}\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k|y-y_s|\right)\phi(y_s)\,\mathrm{d}y_s \\
&\approx \sum_{n=1}^{N}\phi(y_n)\frac{\mathrm{i}}{4}\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k|y-y_s|\right)\,\mathrm{d}y_s.
\end{aligned}
$$

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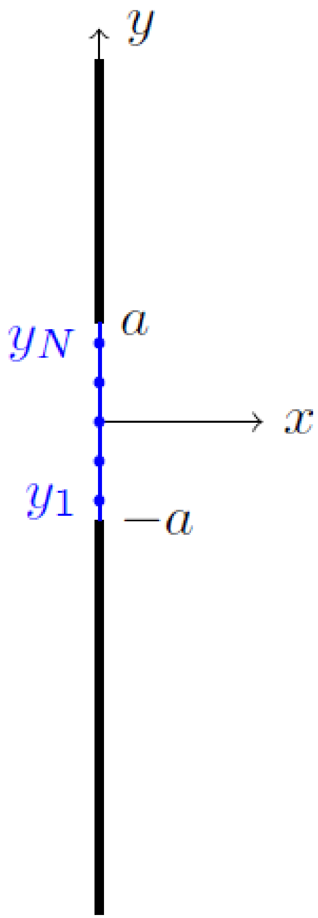

For $-a < y < a$, where $h = (2a)/N$, $y_n = -a + (n-0.5)h$,

$$\begin{aligned} 1 &= \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k|y-y_s|\right)\phi(y_s)\,\mathrm{d}y_s \\ &= \sum_{n=1}^{N}\frac{\mathrm{i}}{4}\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k|y-y_s|\right)\phi(y_s)\,\mathrm{d}y_s \\ &\approx \sum_{n=1}^{N}\phi(y_n)\frac{\mathrm{i}}{4}\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k|y-y_s|\right)\mathrm{d}y_s. \end{aligned}$$

To determine $\phi(y_1), ..., \phi(y_N)$ **by the collocation method** we enforce this last equation at $y = y_m$, $m = 1, ..., N$, leading to

$$1 = \sum_{n=1}^{N}\phi(y_n)\underbrace{\frac{\mathrm{i}}{4}\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k|y_m-y_s|\right)\mathrm{d}y_s}_{a_{mn}}, \quad m = 1, ..., N.$$

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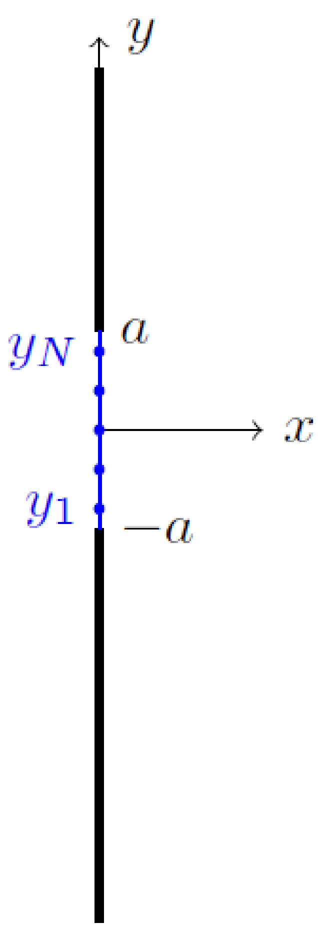


For $-a < y < a$, where $h = (2a)/N$, $y_n = -a + (n - 0.5)h$,

$$1 \quad \approx \quad \sum_{n=1}^{N} \phi(y_n) \frac{\mathrm{i}}{4} \int_{y_n - h/2}^{y_n + h/2} H_0^{(1)}\left(k|y - y_s|\right) \, \mathrm{d}y_s.$$

To determine $\phi(y_1), ..., \phi(y_N)$ **by the Galerkin method** we enforce this last equation by requiring that

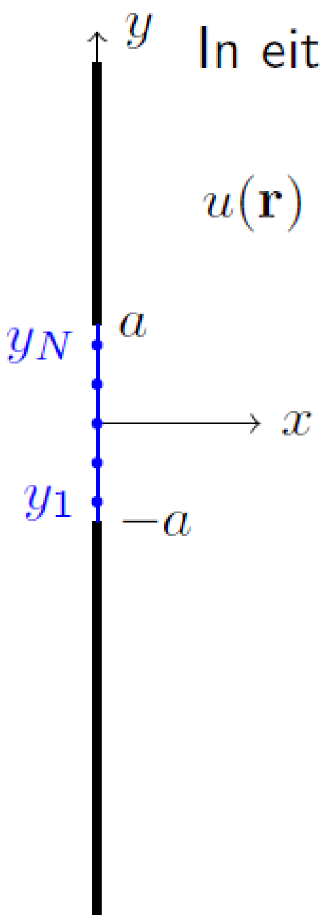
$$\int_{y_m - h/2}^{y_m + h/2} \text{LHS} = \int_{y_m - h/2}^{y_m + h/2} \text{RHS},$$

for $m = 1, ..., N$, leading to

$$h = \sum_{n=1}^{N} \phi(y_n) \underbrace{\frac{\mathrm{i}}{4} \int_{y_m - h/2}^{y_m + h/2} \int_{y_n - h/2}^{y_n + h/2} H_0^{(1)}\left(k|y - y_s|\right) \, \mathrm{d}y_s \, \mathrm{d}y}_{a_{mn}},$$

for $m = 1, ..., N$.

A first Boundary Element Method



In either case, once we have computed $\phi(y_1), \ldots, \phi(y_N)$,

$$\begin{aligned}
u(\mathbf{r}) &= \frac{\mathrm{i}}{4}\int_{-a}^{a} H_0^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s \\
&= \frac{\mathrm{i}}{4}\sum_{n=1}^{N}\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)\phi(y_s)\,\mathrm{d}y_s \\
&\approx \frac{\mathrm{i}}{4}\sum_{n=1}^{N}\phi(y_n)\int_{y_n-h/2}^{y_n+h/2} H_0^{(1)}\left(k\sqrt{x^2+(y-y_s)^2}\right)\mathrm{d}y_s \\
&\approx \frac{h\mathrm{i}}{4}\sum_{n=1}^{N}\phi(y_n)H_0^{(1)}\left(k\sqrt{x^2+(y-y_n)^2}\right).
\end{aligned}$$

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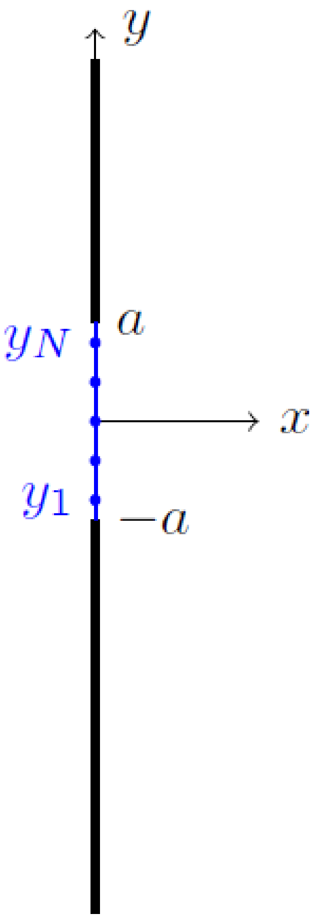
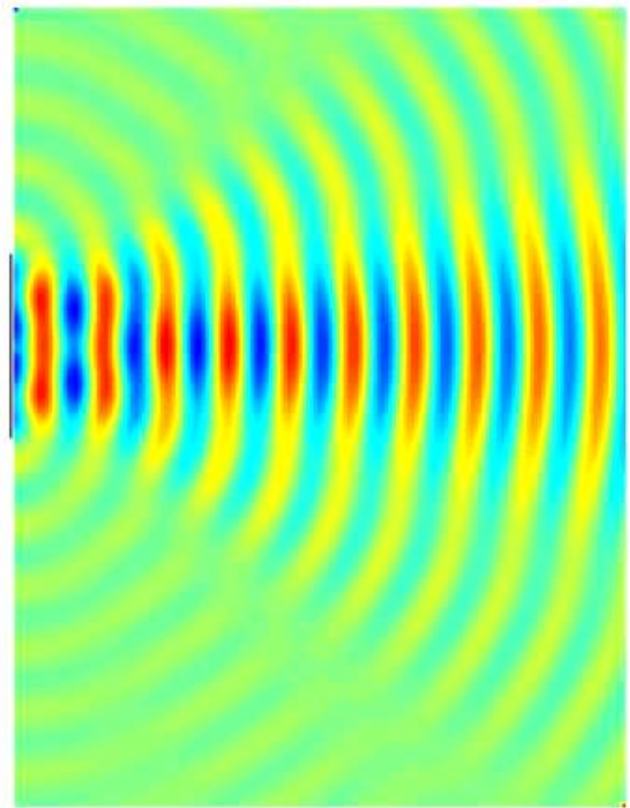
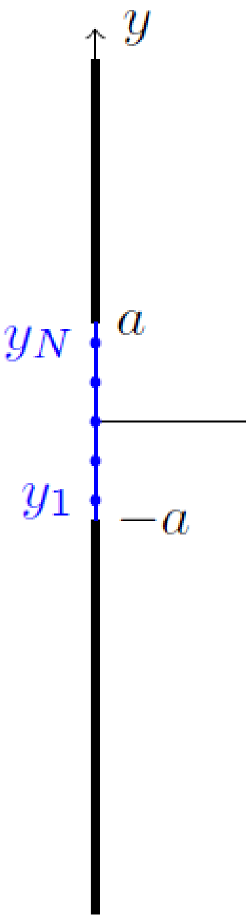
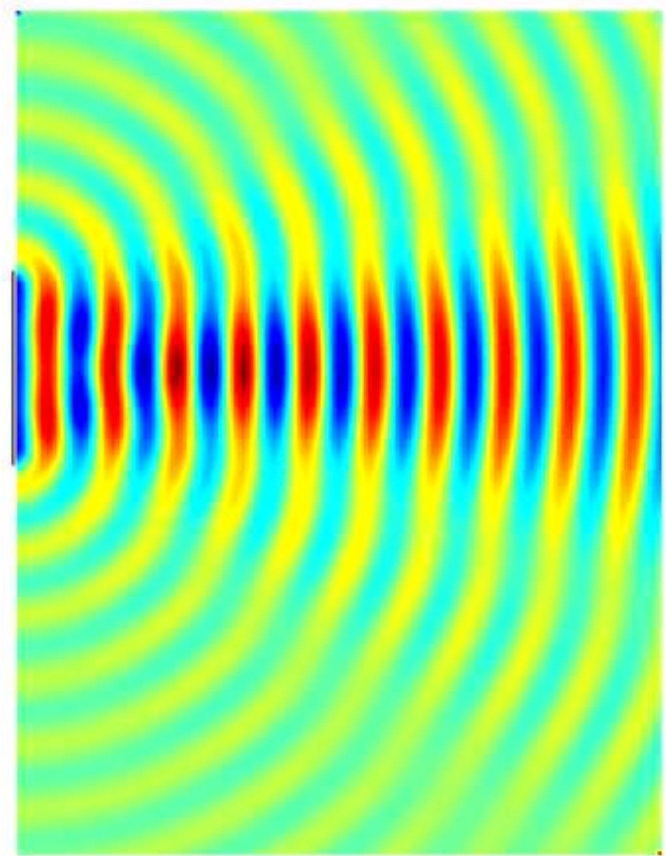


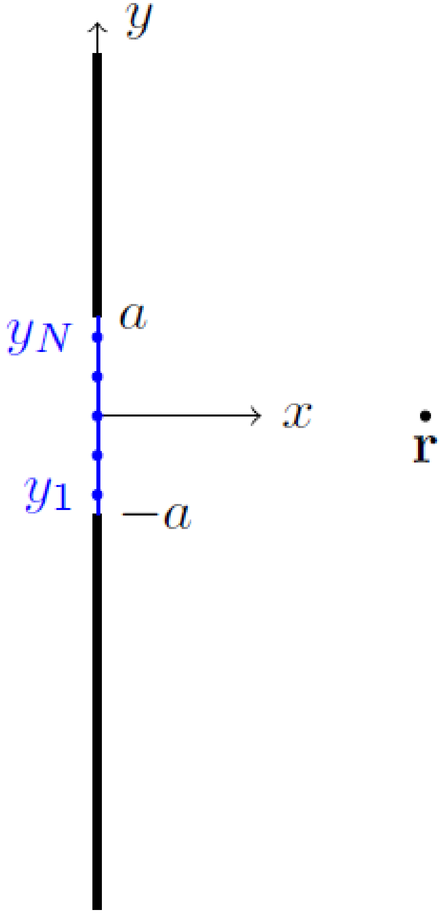


A first Boundary Element Method





How does the accuracy of this BEM depend on h?



Computing $u(\mathbf{r})$ via approximate collocation method at $\mathbf{r} = (5, 0)$ when $\lambda = 1$ and $2a = 3$.

N	λ/h	Relative error	dB error
9	3	15.763%	1.27
18	6	4.169%	0.35
36	12	1.634%	0.14
72	24	0.703%	0.06
Kirchhoff	–	17.68%	0.44

At least 6-10 "degrees of freedom per wavelength", the value of λ/h, recommended for accurate results.

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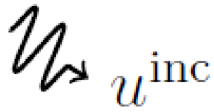
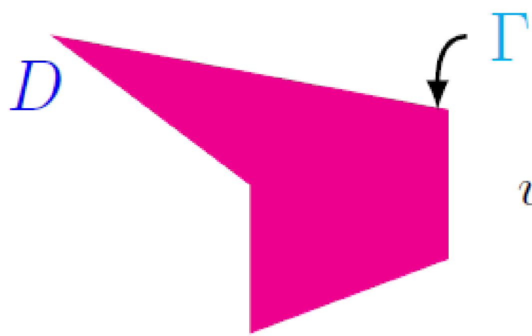
6. Further reading

Acoustic Scattering by an Obstacle and Green's 3rd Identity



$$\Delta u + k^2 u = 0$$



$u - u^{\text{inc}}$ satisfies Sommerfeld R.C.

Theorem For $\mathbf{r}$ in D

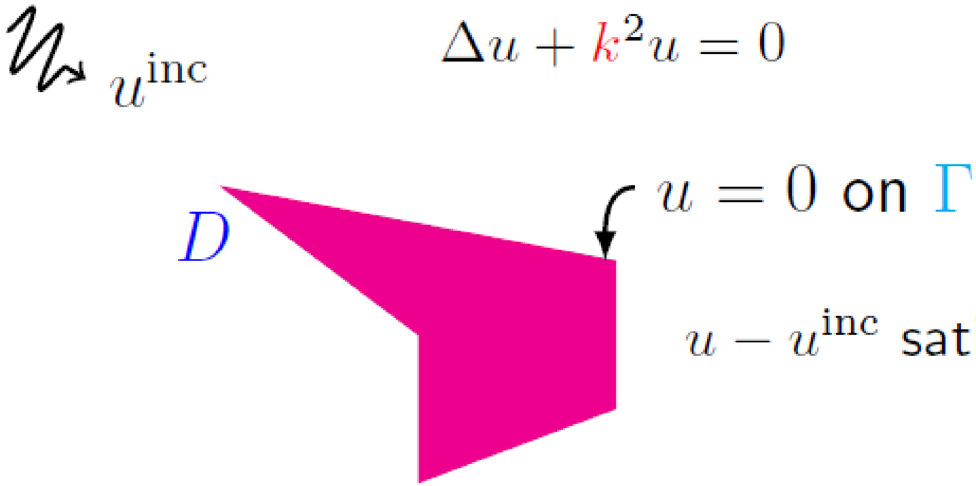
$$u(\mathbf{r}) = u^{\text{inc}}(\mathbf{r}) + \int_{\Gamma} \left(\frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r}, \mathbf{r}_s) - u(\mathbf{r}_s)\frac{\partial \Phi(\mathbf{r}, \mathbf{r}_s)}{\partial n(\mathbf{r}_s)} \right) ds(\mathbf{r}_s),$$

where

$$\Phi(\mathbf{r}, \mathbf{r}_s) = \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_s|) & \text{(2D)}, \\ \dfrac{1}{4\pi} \dfrac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r} - \mathbf{r}_s|}, & \text{(3D)}. \end{cases}$$

Acoustic Scattering by an Obstacle and Green's 3rd Identity

$$\Delta u + k^2 u = 0$$



$u - u^{\text{inc}}$ satisfies S.R.C.

Theorem For $\mathbf{r}$ in D

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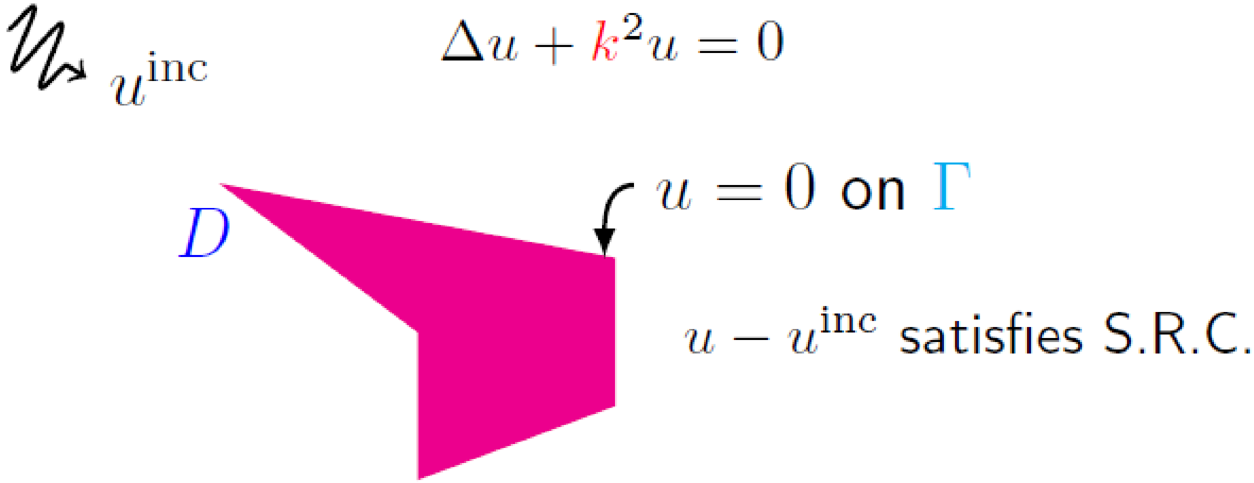
N.B. We only need the **Cauchy data** $u, \dfrac{\partial u}{\partial n}$ on Γ to compute u in D. These can be obtained from **boundary condition** + **boundary integral equation** on Γ.

Acoustic Scattering by an Obstacle and Green's 3rd Identity



Theorem For $\mathbf{r}$ in D

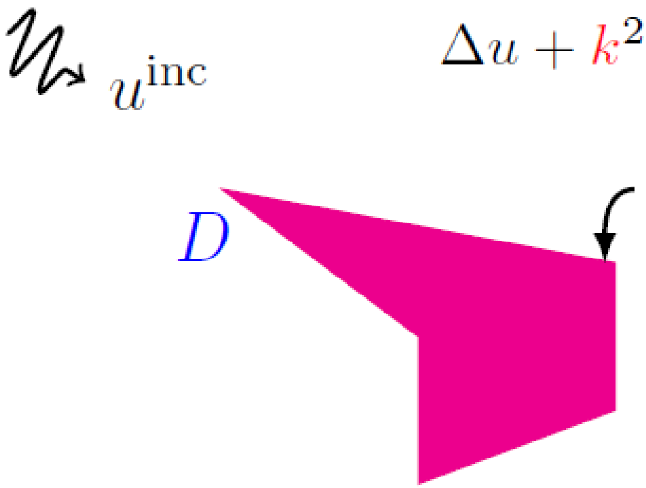
$$u(\mathbf{r}) = u^{\mathrm{inc}}(\mathbf{r}) + \int_\Gamma \frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r}, \mathbf{r}_s)\, ds(\mathbf{r}_s).$$

N.B. We only need the **Cauchy data** $u, \dfrac{\partial u}{\partial n}$ on Γ to compute u in D.

These can be obtained from **boundary condition** + **boundary integral equation** on Γ.

Acoustic Scattering by an Obstacle: Boundary Integral Equation

$$\Delta u + k^2 u = 0$$



$u = 0$ on Γ

$u - u^{\rm inc}$ satisfies S.R.C.

Theorem For $\mathbf{r}$ on Γ

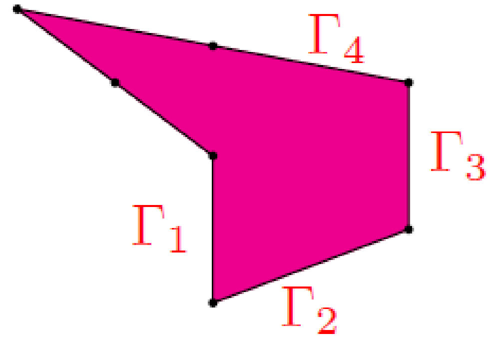
$$0 = u^{\rm inc}(\mathbf{r}) + \int_\Gamma \frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r},\mathbf{r}_s)\, ds(\mathbf{r}_s),$$

the **boundary integral equation** that we solve by the **BEM** to determine $\frac{\partial u}{\partial n}$.

Acoustic Scattering by an Obstacle: Boundary Element Method

u^{inc}

$$\Delta u + k^2 u = 0$$



To solve

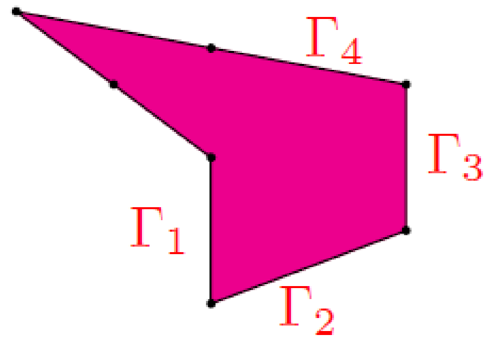
$$0 = u^{\mathrm{inc}}(\mathbf{r}) + \int_{\Gamma} \frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r}, \mathbf{r}_s)\, ds(\mathbf{r}_s),$$

by the BEM:

1. Divide Γ up into N pieces $\Gamma_1, \ldots, \Gamma_N$ with diameter small compared to the wavelength – the **boundary elements**.

Acoustic Scattering by an Obstacle: Boundary Element Method

u^{inc} $\qquad \Delta u + k^2 u = 0$



To solve

$$0 = u^{\text{inc}}(\mathbf{r}) + \sum_{n=1}^{N} \int_{\Gamma_n} \frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r}, \mathbf{r}_s)\, ds(\mathbf{r}_s),$$

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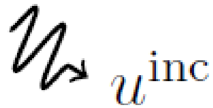

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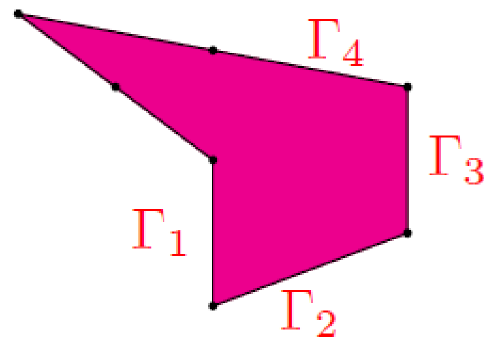


To solve

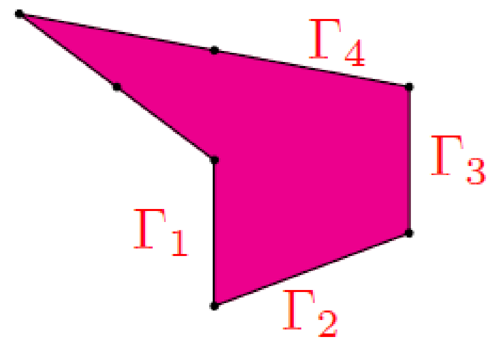
$$0 = u^{\text{inc}}(\mathbf{r}) + \sum_{n=1}^{N} \int_{\Gamma_n} \frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r}, \mathbf{r}_s)\, ds(\mathbf{r}_s),$$

by the BEM:

2. Approximate the unknown function $\phi := \frac{\partial u}{\partial n}$ by a constant ϕ_n on element Γ_n.

Acoustic Scattering by an Obstacle: Boundary Element Method

u^{inc} $\qquad\qquad \Delta u + k^2 u = 0$



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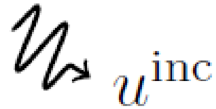

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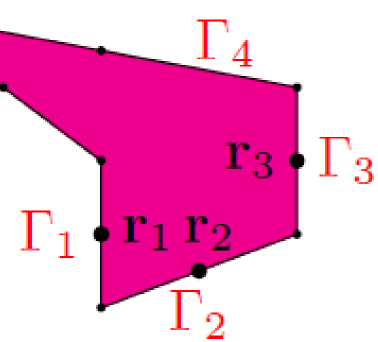

To solve

$$0 = u^{\text{inc}}(\mathbf{r}) + \sum_{n=1}^{N} \phi_n \int_{\Gamma_n} \Phi(\mathbf{r}, \mathbf{r}_s)\, ds(\mathbf{r}_s),$$

by the BEM:

3. Determine $\phi_1, \ldots, \phi_N$ by (in the **collocation version**) enforcing the above equation at the midpoint $\mathbf{r}_m$ of element Γ_m, for $m = 1, \ldots, N$.

Acoustic Scattering by an Obstacle: Boundary Element Method

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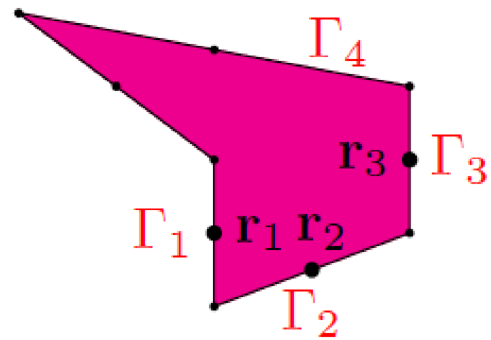

For $m = 1, \ldots, N$,

$$0 = u^{\text{inc}}(\mathbf{r}_m) + \sum_{n=1}^{N} \phi_n \underbrace{\int_{\Gamma_n} \Phi(\mathbf{r}_m, \mathbf{r}_s)\, ds(\mathbf{r}_s)}_{a_{mn}},$$

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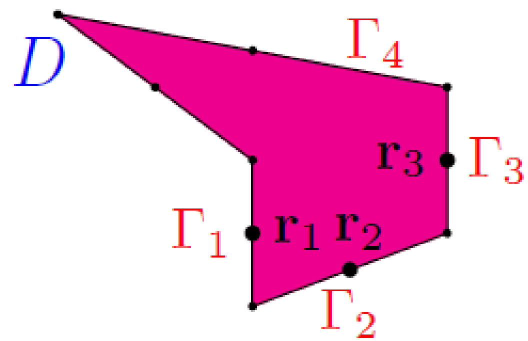
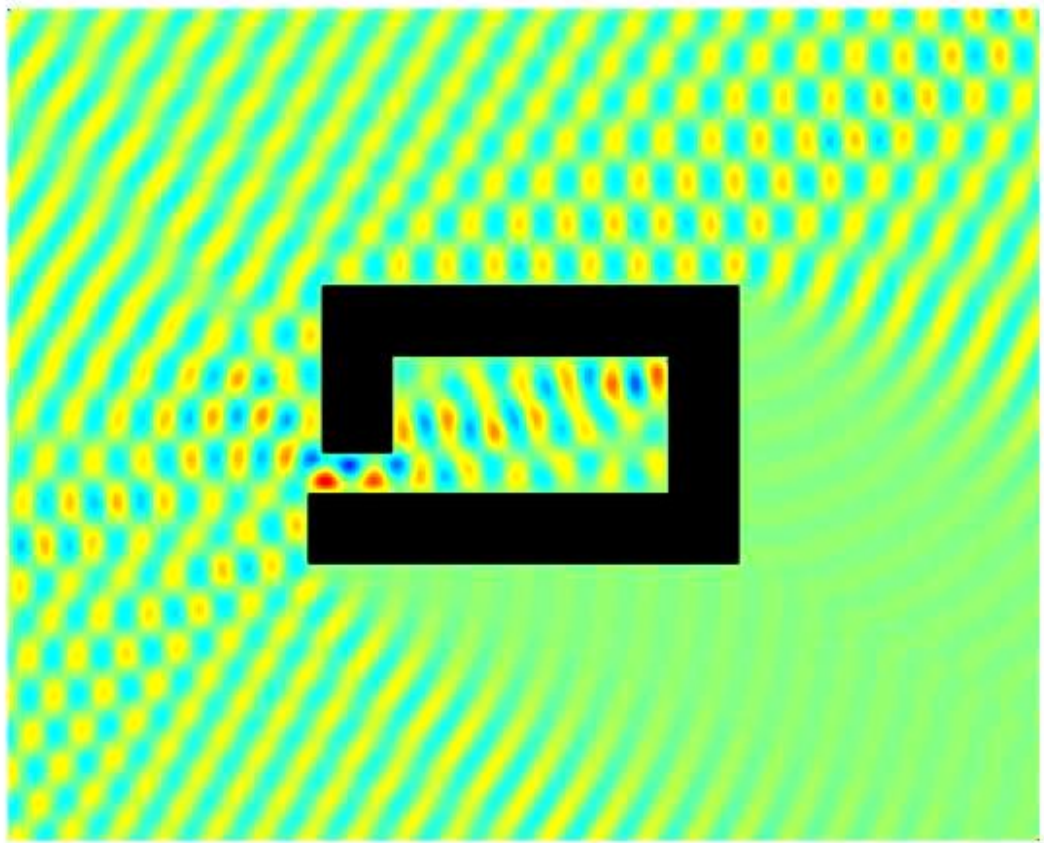


4. Determine $u(\mathbf{r})$ at any desired points in D using

$$\begin{aligned} u(\mathbf{r}) &= u^{\mathrm{inc}}(\mathbf{r}) + \int_{\Gamma} \frac{\partial u}{\partial n}(\mathbf{r}_s)\Phi(\mathbf{r},\mathbf{r}_s)\, ds(\mathbf{r}_s) \\ &\approx u^{\mathrm{inc}}(\mathbf{r}) + \sum_{n=1}^{N} \phi_n \int_{\Gamma_n} \Phi(\mathbf{r},\mathbf{r}_s)\, ds(\mathbf{r}_s). \end{aligned}$$

Acoustic Scattering by an Obstacle: Boundary Element Method

Example 2D simulation. Total length of boundary is $16.6\mathrm{m}$, $\lambda = 0.25\mathrm{m}$, $N = 553$, so 10 elements per wavelength.



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How practical is this?

1. **Restrictions**: mainly useful in homogeneous media (though piecewise constant media possible). Boundaries/interfaces to be discretised must not be too large compared to the wavelength – see 2 and 3.

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2. Main cost is to assemble an $N \times N$ matrix, and solve N equations in N unknowns: cost proportional to N^2 in storage and N^3 in computation time for direct solve, but **fast multipole methods** and **preconditioned iterative solvers, e.g. GMRES** bring these down to $N \log N$ and $N_{\mathrm{Iter}} N \log N$, where N_{Iter} is the number of iterations, respectively, and make $N = 10^6 - 10^7$ feasible.

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3. **How big an N do we need?** Using elements of diameter $\lambda/10$, assuming wavelength $\lambda = 0.343\mathrm{m}$, corresponding to $f = 1000\mathrm{Hz}$ in air, we can discretise with $N = 10^6$: $34\mathrm{km}$ in 2D; an area of $1000\mathrm{m}^2$ in 3D. In 3D this is problematic, e.g. for frequency range for scattering by submarine in underwater acoustics.

How practical is this?

4. Writing simple 2D and 3D code is rather easy – see the next slide – but writing code that achieves fast solves with low storage with good user interfaces is really hard: but see next week's webinar **Boundary element methods in practice: Algorithms, Computations, and Acceleration** by **Prof Timo Betcke**, UCL, including the **open source code BEM++**.

EXAMPLE APPLICATIONS FROM RECENT (2020) PAPERS

- **A fast BEM procedure using the Z-transform and high-frequency approximations for large-scale 3D transient wave problems,** Damien Mavaleix-Marchessoux, Marc Bonnet, Stéphanie Chaillat, Bruno Leblé, Preprint from https://hal.archives-ouvertes.fr/hal-02515371/document

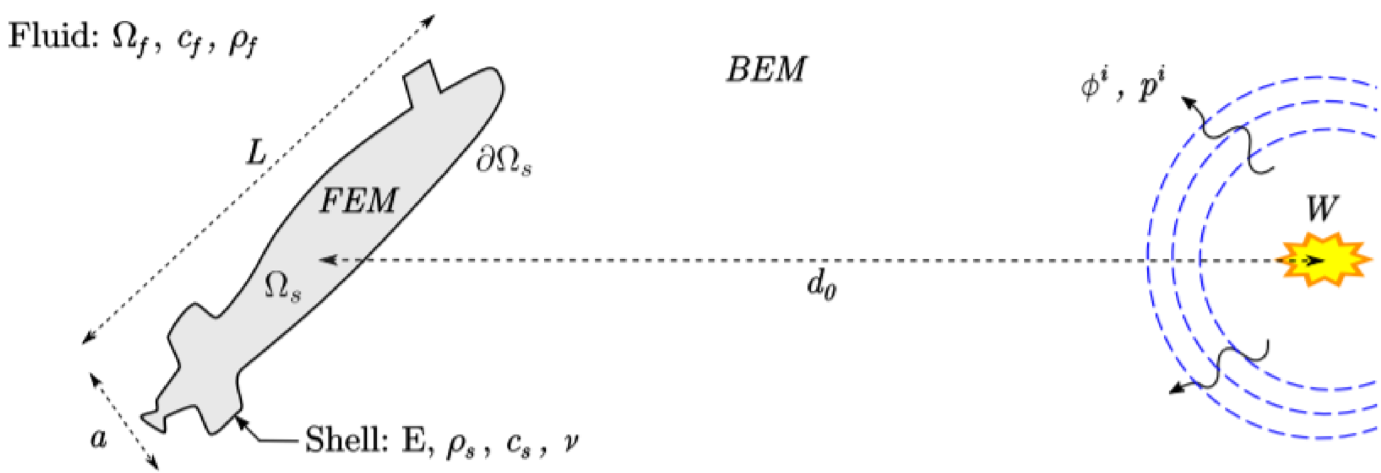

Figure 1: *Submarine experiencing a remote underwater blast.*

EXAMPLE APPLICATIONS FROM RECENT (2020) PAPERS

- **Investigation of radiation damping in sandwich structures using finite and boundary element methods and a nonlinear eigensolver,** Suhaib Koji Baydouna) and Steffen Marburg, Journal of the Acoustical Society of America 147, 2020 (2020); https://doi.org/10.1121/10.0000947

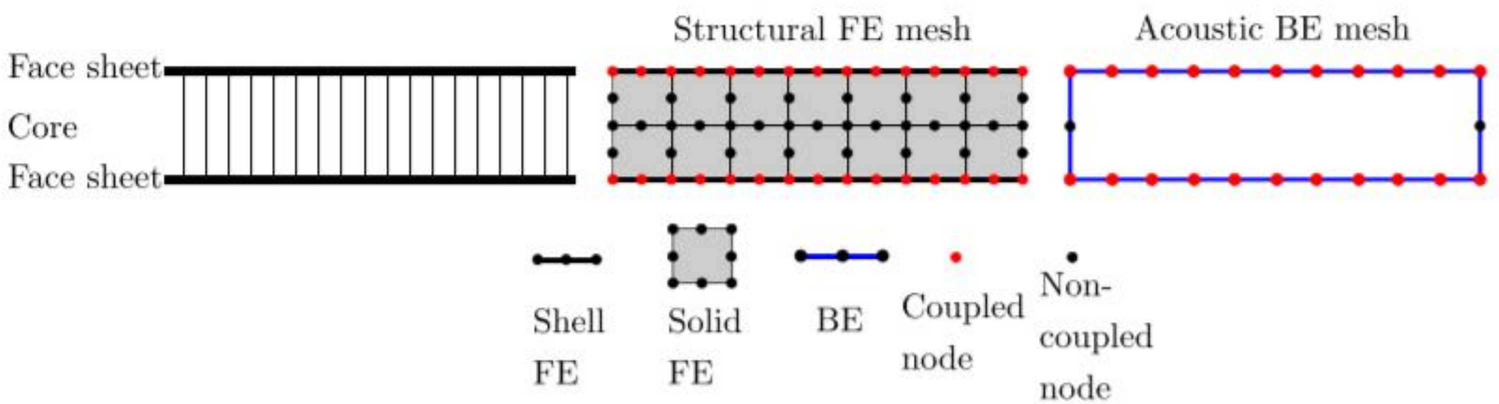

FIG. 3. (Color online) Cross-sectional schematic illustrating the numerical modeling of a (non-baffled) sandwich panel and the surrounding acoustic field. The structural FE mesh is coupled to the closed acoustic BE mesh via non-coincident nodes on the radiating surface.

EXAMPLE APPLICATIONS FROM RECENT (2020) PAPERS

- **Vibro-acoustic Response in Vehicle Interior and Exterior Using Multibody Dynamic Systems Due To Cleat Impacts,** Myeong Jae Han, Chul Hyung Lee and Tae Won Park, International Journal of Automotive Technology, Vol. 21, No. 3, pp. 591-602 (2020); https://link.springer.com/content/pdf/10.1007/s12239-020-0056-1.pdf

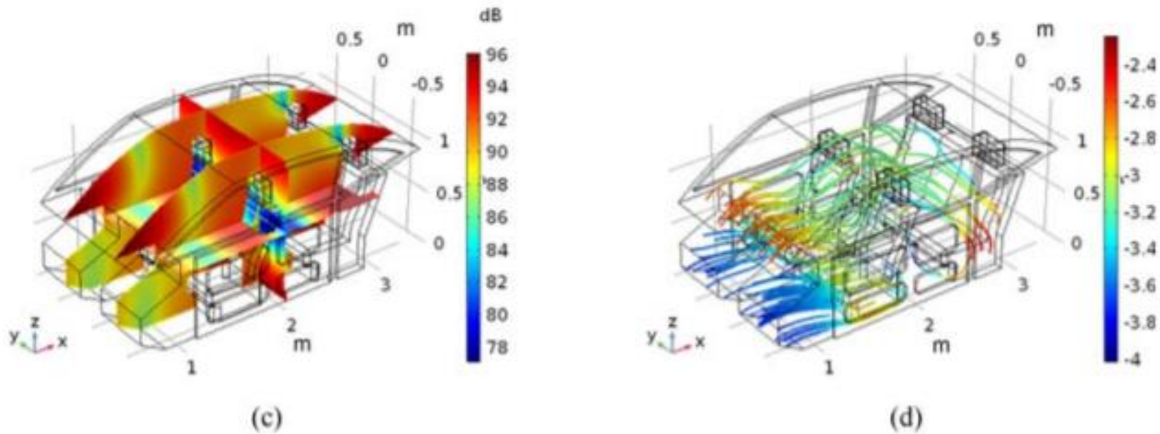

Figure 15. Acoustic characteristics of the vehicle interior: (a) Input boundary surface for normal acceleration; (b) Sound pressure; (c) SPL; (d) Sound intensity.

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FURTHER READING

- My lecture notes, exercises, and simple downloadable Matlab code, with Steve Langdon on **Boundary Elements in Acoustics**, aimed at PhD students in acoustics
  http://www.personal.reading.ac.uk/~sms03snc/smart_numerics.html
- Many books on boundary element method in general: fewer on acoustics:

❖ The Boundary Element Method: Vol 1. Applications in Thermo-fluids and Acoustics, Luiz Wrobel, Wiley 2007

❖ Computational Acoustics of Noise Propagation in Fluids - Finite and Boundary Element Methods, Steffen Marburg & Bodo Nolte (Eds.), Springer 2008

❖ Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements, Olaf Steinbach, Springer 2008

❖ Boundary Element Methods, Stefan Sauter & Christoph Schwab, Springer 2011

FURTHER READING

My own review papers:

- The Boundary Element Method in Outdoor Noise Propagation, *Proceedings of the Institute of Acoustics* **19,** 27-50 (1997).
- Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering, with I G Graham, S Langdon, & E A Spence *Acta Numerica*, **21**, 89-305 (2012).
- Acoustic scattering: high frequency boundary element methods and unified transform methods, with S Langdon, in *Unified Transform for Boundary Value Problems: Applications and Advances*, A S Fokas & B Pelloni (editors), SIAM, 2015.

FURTHER READING

And remember next’s week’s Webinar!
https://acoustics.ac.uk/events/webinar/

Webinar – Boundary element methods in practice: Algorithms, Computations, and Acceleration

Posted in Computational Acoustics, Events, Mathematical Analysis in Acoustics

- May 13, 2020
- 15:00 – 17:00
- Webinar via Zoom
- Website

Webinar “Boundary element methods in practice: Algorithms, Computations, and Acceleration”, Professor Timo Betcke, UCL