On Spectral Inclusion Sets and Computing the Spectra and Pseudospectra of Bounded Linear Operators

Simon Chandler-Wilde

August 2024, IWOTA, University of Kent

These slides available at **tinyurl.com/5642d82j**

...with the help of...

This talk is based on work in *J. Spectr. Theor.* (2024)

`https://ems.press/journals/jst/articles/14297880`

with

- Marko Lindner, TU Hamburg, Germany
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I'll mention at the end follow-on work with Marko, including

arXiv:2408.03883

and our joint work in progress with Christian Seifert (TU Hamburg): see also **Marko's talk** in the session **Spectral Problems and Computation** in this room at **3pm.**

What is this talk about?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
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Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x,x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x,y) = \sum_{j\in\mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j\in\mathbb{Z}} |x_j|^2.$$

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If E, Y are Hilbert spaces, $L(E, Y)$ is the set of **bounded linear operators** from E to Y. The **norm** and **lower norm** of $A \in L(E, Y)$ are

$$\|A\| := \sup_{x \in E \setminus \{0\}} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \nu(A) := \inf_{x \in E \setminus \{0\}} \frac{\|Ax\|}{\|x\|}.$$

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For $A \in L(E, Y)$, $A^* \in L(Y, E)$ is its **adjoint**, and $A \in L(E) := L(E, E)$ is **normal** if $AA^* = A^*A$.

The lower norm and invertibility

Recall $\nu(A) := \inf_{x \in E \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$. If $A \in L(E)$ then

$$A \text{ is not invertible} \quad \Leftrightarrow \quad \mu(A) := \min(\nu(A), \nu(A^*)) = 0,$$

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With $\|A^{-1}\|^{-1} := 0$ if A is not invertible,

$$\mu(A) = \|A^{-1}\|^{-1}, \qquad \text{for all } A \in L(Y).$$

For $A \in L(E)$ the **spectrum** of A is

$$\mathrm{Spec}\,A := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\},$$

where

$$\mu(A - \lambda I) := \min(\nu(A - \lambda I), \nu((A - \lambda I)^*)) = \|(A - \lambda I)^{-1}\|^{-1}.$$

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For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

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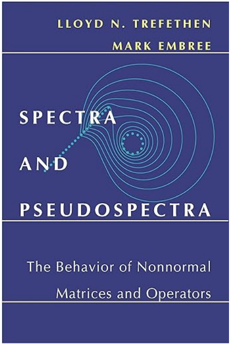
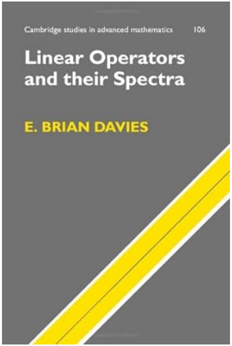
More on pseudospectra: Trefethen & Embree 2005, Davies 2007





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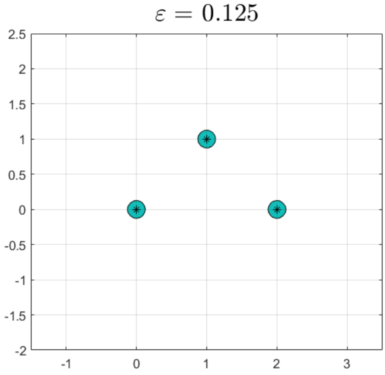
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 + \mathrm{i} \end{bmatrix}$$



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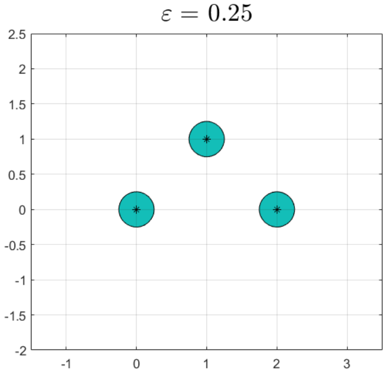

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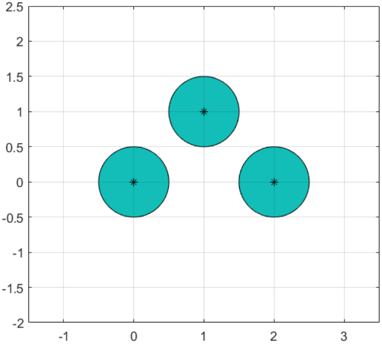

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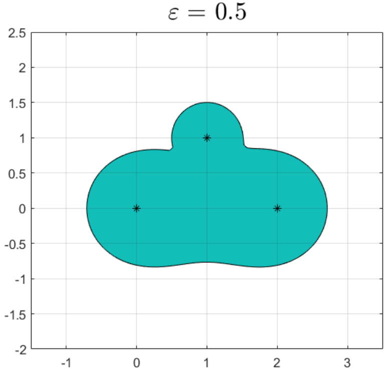
$$A = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{15} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{1}{6} + \frac{\mathrm{i}}{10} \\ 0 & 0 & 1+\mathrm{i} \end{bmatrix}$$



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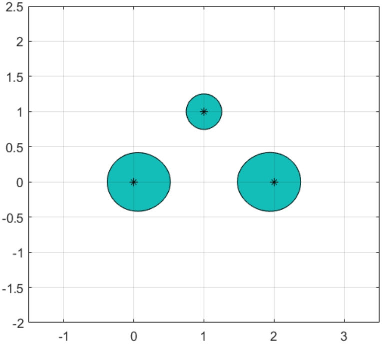

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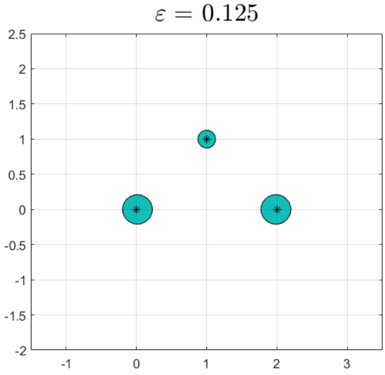

Hausdorff convergence of sets

Let $\mathbb{C}^C :=$ set of non-empty **compact** subsets of $\mathbb{C}$

For $S, T \in \mathbb{C}^C$ let

$$d(S,T) := \inf\{\varepsilon \geq 0 : S \subset T + \varepsilon\mathbb{D} \text{ and } T \subset S + \varepsilon\mathbb{D}\},$$

so $d(\cdot,\cdot)$ is the **Hausdorff metric** on $\mathbb{C}^C$.

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Corollary. If $\varepsilon_1 > \varepsilon_2 > \ldots > 0$, in which case $\varepsilon_n \to \varepsilon \geq 0$ as $n \to \infty$, then

$$\mathrm{Spec}_{\varepsilon_n} A \to \mathrm{Spec}_\varepsilon A \qquad \text{N.B.} \quad \mathrm{Spec}_0 A = \mathrm{Spec}\, A.$$

Matrix representation of A

Suppose $(e_j)_{j\in\mathbb{Z}}$ is an orthonormal basis for E and $A \in L(E)$. Then the **matrix representation** of A is $[A] = [a_{ij}]_{i,j\in\mathbb{Z}}$, where

$$a_{ij} = (Ae_j, e_i), \qquad i, j \in \mathbb{Z},$$

and $\mathrm{Spec}\, A = \mathrm{Spec}\, [A]$, $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon [A]$, $\varepsilon > 0$, where $[A] \in L(\ell^2)$ is defined by

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We will say that A is **banded** with **bandwidth** $w \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ if $a_{ij} = 0$ for $|i - j| > w$.

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We will say that A is **banded** with **bandwidth** $w \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ if $a_{ij} = 0$ for $|i - j| > w$.

We will say that A is **band-dominated** if there exists a sequence $(A_n) \subset L(E)$ such that each A_n is banded and $\|A - A_n\| \to 0$ as $n \to \infty$.

What is this talk about?

Question. Given a **band-dominated bi-infinite matrix** $A \in L(E)$, with $E = \ell^2(\mathbb{Z})$, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

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The (key) tridiagonal case

Let's consider first bi-infinite matrices of the form

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_2 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$ and $\gamma = (\gamma_i)$ are bounded sequences of complex numbers.

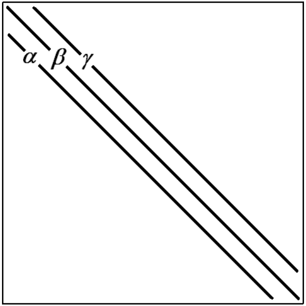
$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_2 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

Task

Compute **inclusion sets for spectrum and pseudospectra** of $A \in L(\ell^2) = L(\ell^2(\mathbb{Z}))$.

Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:



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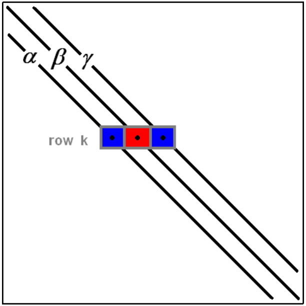

For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

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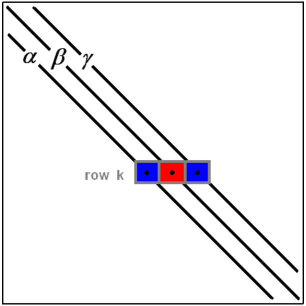

For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

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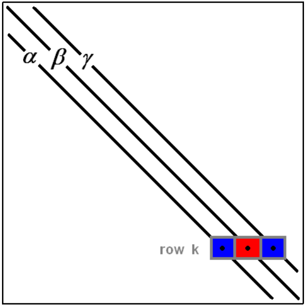

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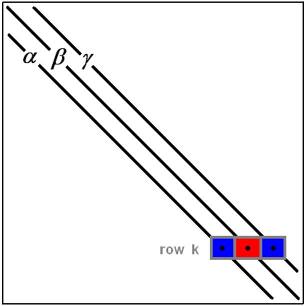

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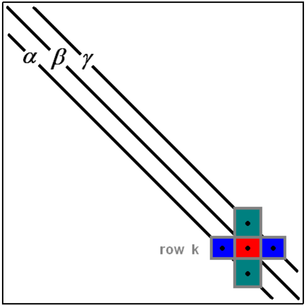

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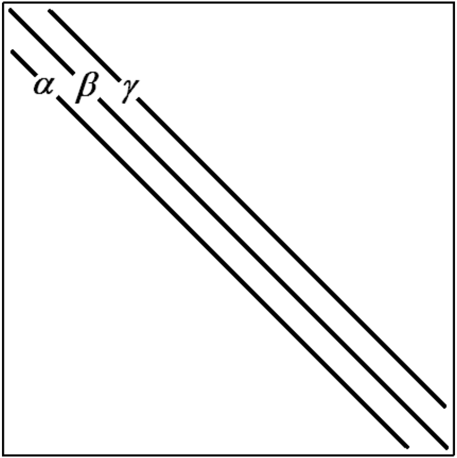
Look at (pseudo)spectra of the **finite principal submatrices** of A:



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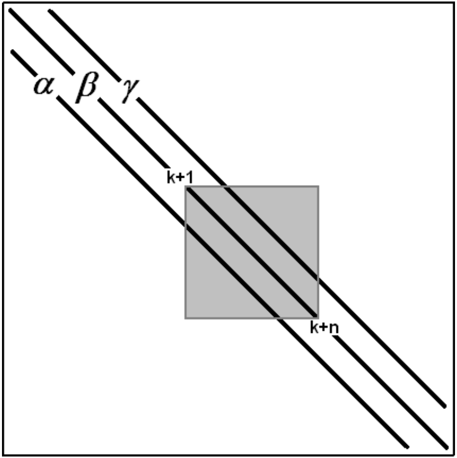

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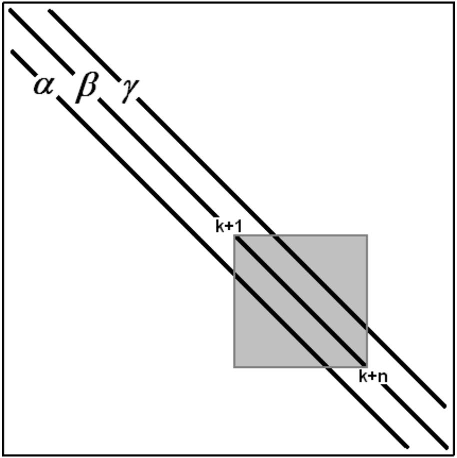

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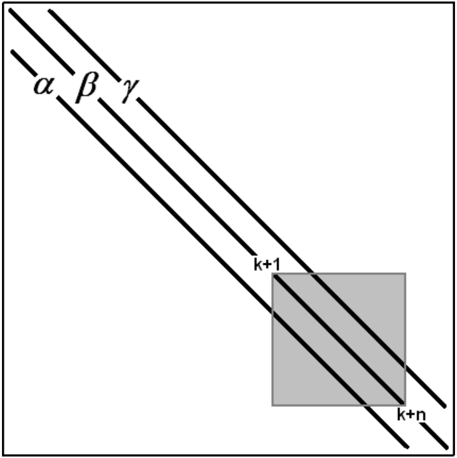
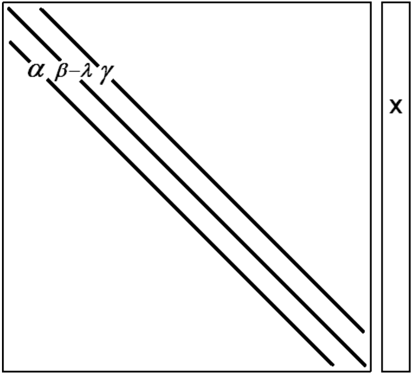


τ method: finite principal submatrices

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



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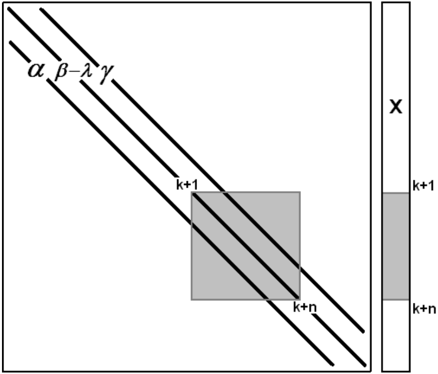

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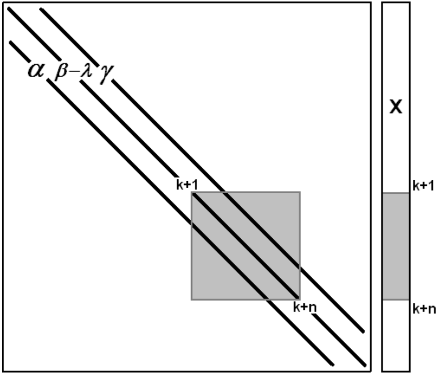

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$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

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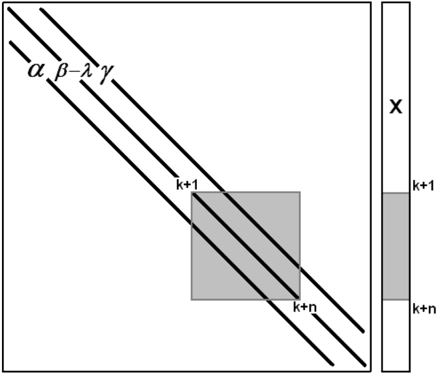

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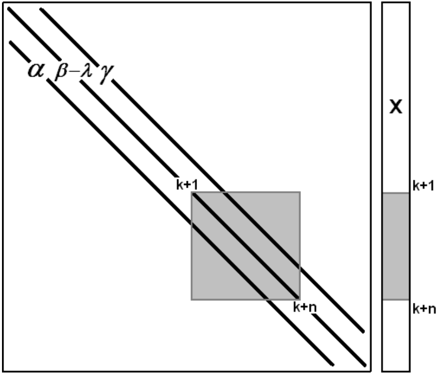

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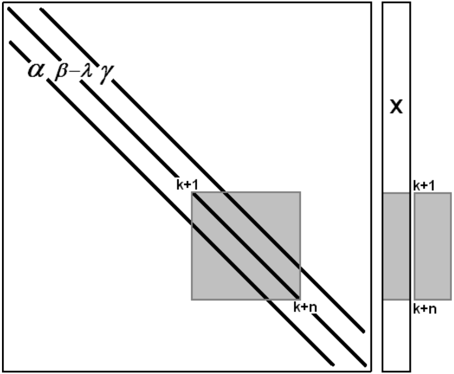

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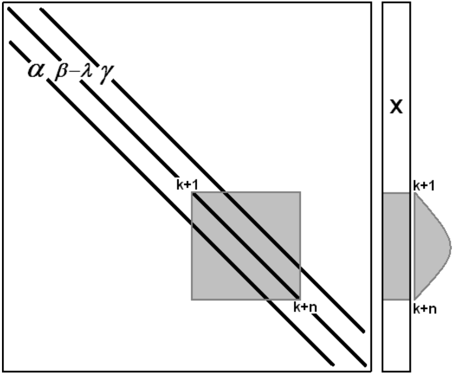

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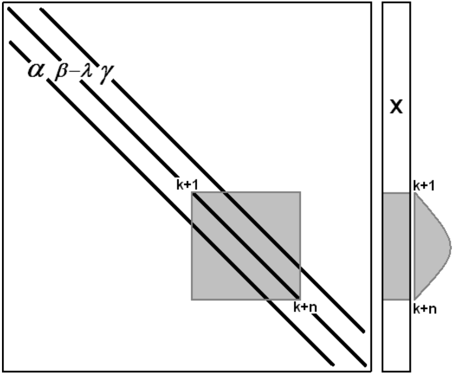

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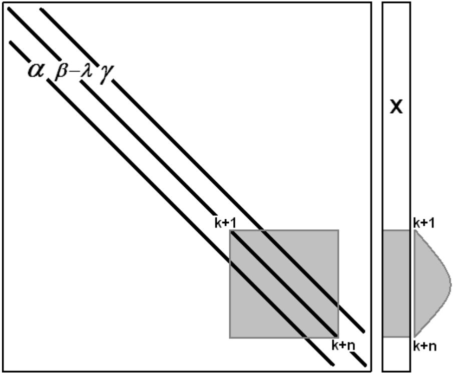

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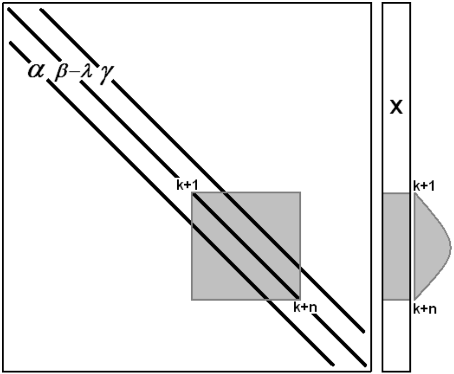

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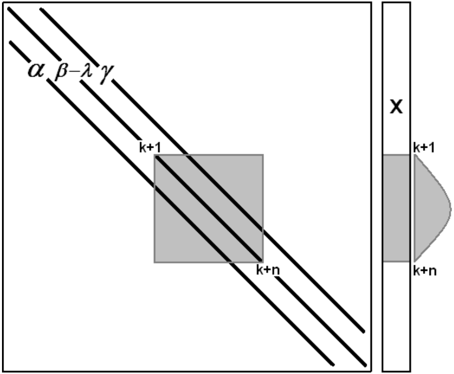

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So we get

Inclusion Set

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}, \quad \varepsilon \geq 0,$$

where

$$\varepsilon_n \;=\; 2\sin\left(\frac{\pi}{2(n+2)}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty),$$

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τ method: finite principal submatrices

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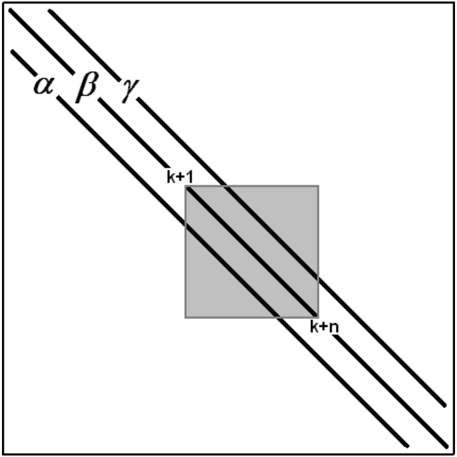
so $\varepsilon_n = O(n^{-1})$ as $n \to \infty$. Putting $n = 1$ and $\varepsilon = 0$ we recover Gershgorin:

$$\mathrm{Spec}\, A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_1} A_{1,k}} = \overline{\bigcup_{k\in\mathbb{Z}} (a_{k,k} + (\|\alpha\|_\infty + \|\gamma\|_\infty)\mathbb{D})}.$$

π method: **periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised" (cf. Colbrook 2020, which uses **single large periodised finite section**)



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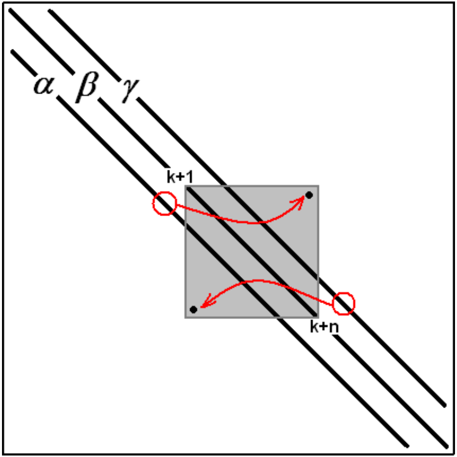

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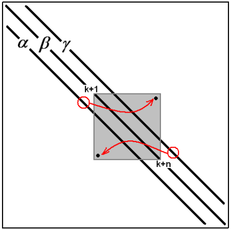


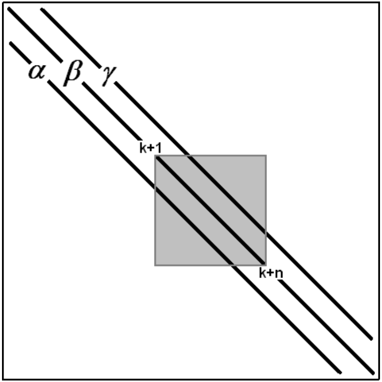
very similar computations show that

$$\mathrm{Spec}_\varepsilon A \ \subset\ \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon'_n} A^{\mathrm{per}}_{n,k}}, \qquad \varepsilon \geq 0,$$

$$\text{with} \qquad \varepsilon'_n \;=\; 2\sin\left(\frac{\pi}{2n}\right)\left(\|\alpha\|_\infty + \|\gamma\|_\infty\right).$$

Here is another idea: τ_1 method

Instead of



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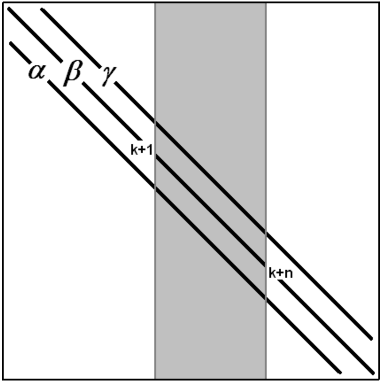
We do a **"one-sided"** truncation.



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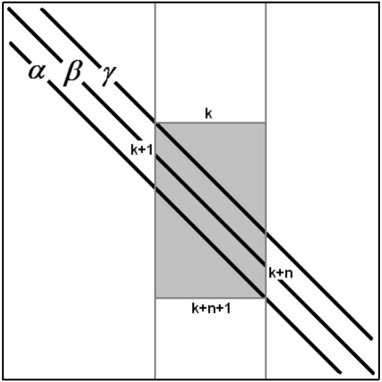
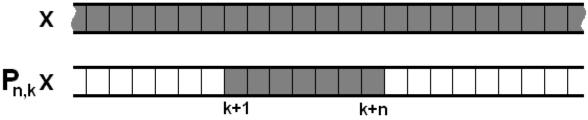


I.e., we work with **rectangular** finite submatrices.

This is motivated by work of Davies 1998, Davies & Plum 2004, and Hansen 2008, 2011, in which A is approximated by a **single large rectangular finite section**.

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $P_{n,k} : \ell^2 \to \ell^2$ denote the projection

$$(P_{n,k}x)(i) \;:=\; \begin{cases} x(i), & i \in \{k+1, ..., k+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

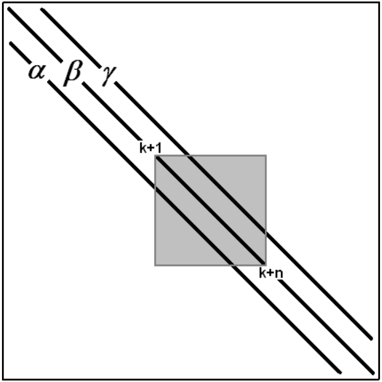


Further, we put

$$E_{n,k} := \operatorname{im} P_{n,k}.$$

τ_1 method: truncations

τ method:



$$P_{n,k}(A - \lambda I)|_{E_{n,k}}$$

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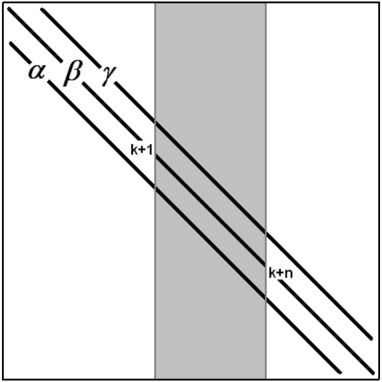

$$(A - \lambda I)|_{E_{n,k}}$$

τ_1 method: τ method revisited

τ method:

$$\lambda \in \mathrm{Spec}_\varepsilon A \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k} = \mathrm{Spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k} A|_{E_{n,k}}\right),$$

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$$\text{i.e.} \quad \min\left(\left(\nu\big(P_{n,k}(A-\lambda I)|_{E_{n,k}}\big), \nu\big(P_{n,k}(A-\lambda I)^*|_{E_{n,k}}\big)\right) \le \varepsilon + \varepsilon_n.$$

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τ_1 idea is just **drop the $P_{n,k}$'s**.

Replace $\mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}$ by

$$\gamma^{n,k}_{\varepsilon+\varepsilon_n}(A) := \{\lambda : \min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \leq \varepsilon + \varepsilon_n\}.$$

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \leq \varepsilon.$$

(Analogue of $\mathrm{Spec}_\varepsilon A_{n,k}$ in the τ method.)

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Then (similarly to the τ and π-method inclusions)

$$\mathrm{Spec}_\varepsilon A \quad\subset\quad \Gamma^n_{\varepsilon+\varepsilon_n''}(A),$$

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But **now we also have** that if $\lambda \in \gamma_\varepsilon^{n,k}(A)$, for some $k \in \mathbb{Z}$, then $\nu(A-\lambda) \leq \nu((A-\lambda I)|_{E_{n,k}}) \leq \varepsilon$ or $\nu((A-\lambda)^*) \leq \varepsilon$

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$$\Gamma_\varepsilon^n(A) \quad\subset\quad \mathrm{Spec}_\varepsilon A.$$

τ_1-method: spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{Spec}_\varepsilon A \qquad \text{and} \qquad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A)$$

we get

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$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A), \quad \varepsilon \geq 0.$$

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Since $\mathrm{Spec}_{\varepsilon+\varepsilon''_n} A \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, we have

$$\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \to \quad \mathrm{Spec}_\varepsilon A$$

τ_1-method: spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \ \subset \ \mathrm{Spec}_\varepsilon A \qquad \text{and} \qquad \mathrm{Spec}_\varepsilon A \ \subset \ \Gamma^n_{\varepsilon+\varepsilon''_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A), \quad \varepsilon \geq 0.$$

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$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \subset \quad \mathrm{Spec}_{\varepsilon+\varepsilon''_n} A, \quad \varepsilon \geq 0.$$

Since $\mathrm{Spec}_{\varepsilon+\varepsilon''_n} A \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, we have

$$\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \to \quad \mathrm{Spec}_\varepsilon A, \quad \text{in particular} \quad \Gamma^n_{\varepsilon''_n}(A) \quad \to \quad \mathrm{Spec}\, A.$$

The shift operator

Let's compute the τ, π, and τ_1 inclusion sets for $\mathrm{Spec}\, A$, i.e.

$$\begin{array}{ll} \tau \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n} A_{n,k}} \\ \pi \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon'_n} A^{\mathrm{per}}_{n,k}} \\ \tau_1 \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}, \end{array}$$

where

$$\gamma^{n,k}_{\varepsilon''_n}(A) = \left\{\lambda \in \mathbb{C} : \min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \le \varepsilon''_n\right\},$$

in the case that A is the **shift operator**, so that
$\alpha = (\ldots, 0, 0, \ldots)$, $\beta = (\ldots, 0, 0, \ldots)$, $\gamma = (\ldots, 1, 1, \ldots)$,

$$\mathrm{Spec}\, A = \mathbb{T} = \{z : |z| = 1\}$$

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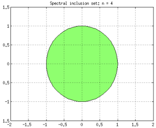
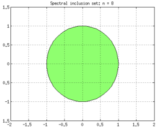
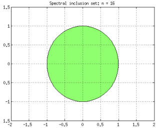
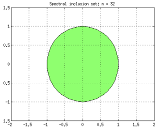
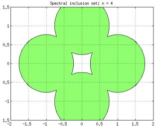
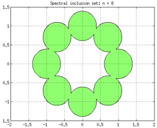
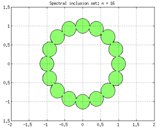
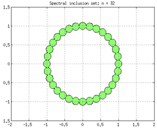
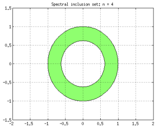
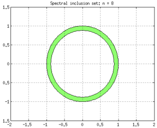
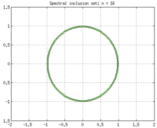
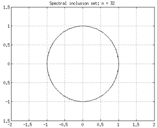
$$\varepsilon_n, \varepsilon'_n, \varepsilon''_n \le 2\sin\left(\frac{\pi}{2n}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty) = 2\sin\left(\frac{\pi}{2n}\right),$$

and the matrices $A_{n,k}$, $k \in \mathbb{Z}$, are all the same!

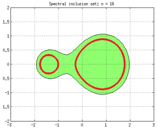
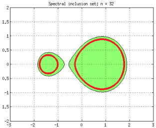
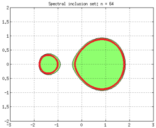
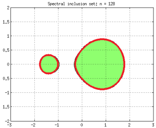
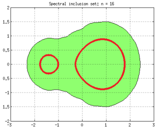
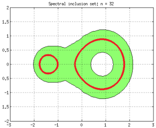
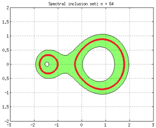
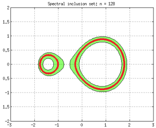
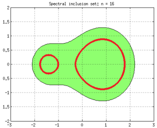
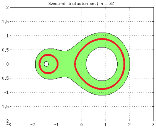
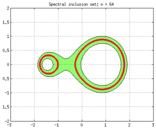
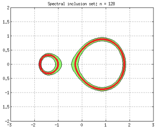
The shift operator

	$n = 4$	$n = 8$	$n = 16$	$n = 32$
τ method				
π method				
τ_1 method				

We now look at a tridiagonal matrix A with 3-periodic diagonals:

1st sub-diagonal $\alpha = (\cdots, 0, 0, 0, \cdots)$
main diagonal $\beta = (\cdots, -\frac{3}{2}, 1, 1, \cdots)$
super-diagonal $\gamma = (\cdots, 1, 2, 1, \cdots)$

3-periodic example

	$n = 16$	$n = 32$	$n = 64$	$n = 128$
τ method				
π method				
τ_1 method				

Let's take stock: what were we trying to do?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
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The missing ingredients

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A pseudoergodic example

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & a_0 & 0 & 1 & \\ & & a_1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where $a = (\cdots, a_0, a_1, \cdots) \in \mathcal{A}^{\mathbb{Z}}$ is **pseudoergodic with respect to the compact set** $\mathcal{A}$ (Davies 2001)

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$$U_n = \bigcup_{\alpha\in\mathcal{A}^{n-1}} \mathrm{Spec}_{\varepsilon_n} A_n(\alpha)$$

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Further, where $\widetilde{\mathcal{A}} \subset \mathcal{A}$ is some finite ε-net for $\mathcal{A}$, with $\varepsilon = 1/n$,

$$U_n = \bigcup_{\alpha\in\mathcal{A}^{n-1}} \mathrm{Spec}_{\varepsilon_n} A_n(\alpha) \approx \widetilde{U}_n := \bigcup_{\alpha\in\widetilde{\mathcal{A}}^{n-1}} \mathrm{Spec}_{\varepsilon_n+1/n} A_n(\alpha)$$

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$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & a_0 & 0 & 1 & \\ & & a_1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad A_n(\alpha) = \begin{pmatrix} 0 & 1 & & & \\ \alpha_1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & 0 & 1 \\ & & & \alpha_{n-1} & 0 \end{pmatrix},$$

where $a = (\cdots, a_0, a_1, \cdots) \in \mathcal{A}^{\mathbb{Z}}$ is **pseudoergodic with respect to the compact set** $\mathcal{A}$ (Davies 2001); i.e., **every** $\mathcal{A}$**-valued finite sequence can be found to arbitrary precision somewhere in** a**.**

If $\mathcal{A} \subset \overline{\mathbb{D}}$ the τ method converges (cf. C-W & Lindner 2016):

$$\mathrm{Spec}\, A \subset \widetilde{U}_n \to \mathrm{Spec}\, A, \quad \text{as} \quad n \to \infty.$$

Further, where $\widetilde{\mathcal{A}} \subset \mathcal{A}$ is some finite ε-net for $\mathcal{A}$, with $\varepsilon = 1/n$,

$$U_n = \bigcup_{\alpha \in \mathcal{A}^{n-1}} \mathrm{Spec}_{\varepsilon_n} A_n(\alpha) \approx \widetilde{U}_n := \bigcup_{\alpha \in \widetilde{\mathcal{A}}^{n-1}} \mathrm{Spec}_{\varepsilon_n + 1/n} A_n(\alpha)$$

$$A = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & a_0 & 0 & 1 & \\ & & a_1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

where $a = (\cdots, a_0, a_1, \cdots) \in \mathcal{A}^{\mathbb{Z}}$ is **pseudoergodic with respect to $\mathcal{A} = \{\pm 1\}$** i.e., **every finite sequence of ± 1's can be found somewhere in a.**

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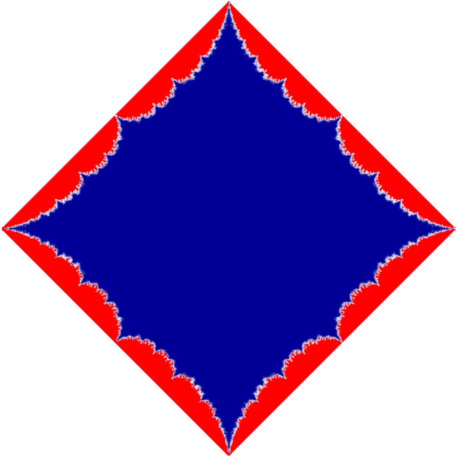
The τ method is convergent:

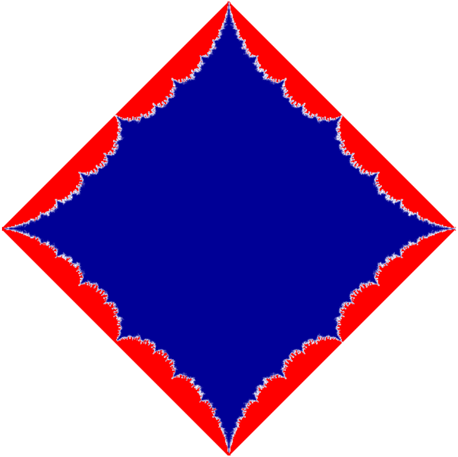
$$\operatorname{Spec} A \subset U_n = \bigcup_{k \in \mathbb{Z}} \operatorname{Spec}_{\varepsilon_n} A_{n,k} \to \operatorname{Spec} A, \quad \text{as} \quad n \to \infty.$$

and the union is finite: 2^{n-1} different matrices $A_{n,k}$.

Upper and lower bounds on $\mathrm{Spec}\, A$: which is sharp?

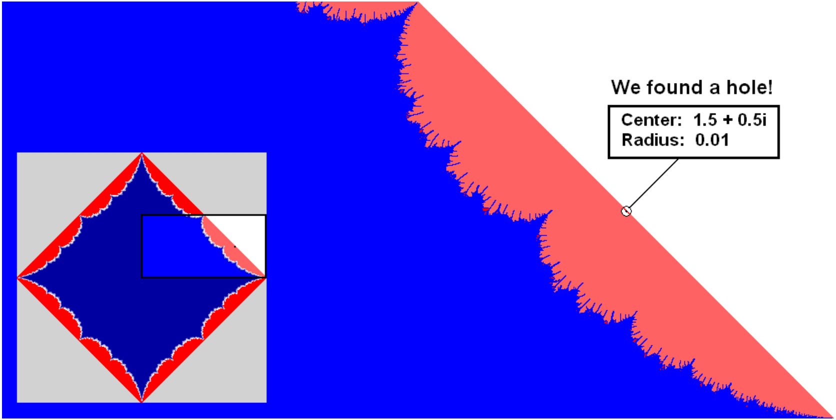


(The square has corners at ± 2 and $\pm 2\mathrm{i}$.)



(The square has corners at ± 2 and $\pm 2\mathrm{i}$.)

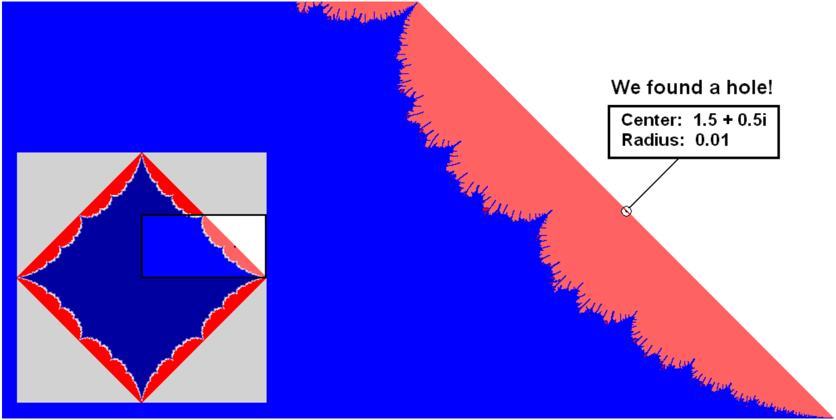
We have $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$ so, if $\lambda \notin \mathrm{Spec}\, A$, then $\lambda \notin U_n$ **for all sufficiently large** n.



$$\lambda = 1.5 + 0.5\mathrm{i} \quad \notin \quad U_{34} \quad \supset \quad \mathrm{Spec}\, A, \qquad \text{so} \qquad \lambda \notin \mathrm{Spec}\, A,$$

so $\mathrm{Spec}\, A$ is a strict subset of the square.

Is $\lambda = 1.5 + 0.5\mathrm{i} \in \operatorname{Spec} A$?



$$\lambda = 1.5 + 0.5\mathrm{i} \;\notin\; U_{34} \;\supset\; \operatorname{Spec} A, \quad \text{so} \quad \lambda \notin \operatorname{Spec} A,$$

so $\operatorname{Spec} A$ is a strict subset of the square. This was a large calculation: we needed to check whether $2^{33} \approx 8.6 \times 10^9$ matrices of size 34×34 were positive definite!

Summary and conclusion

1. For **tridiagonal** A we have derived the τ, π, and τ_1 **inclusion set families** for $\mathrm{Spec}_\varepsilon A$, for $\varepsilon \geq 0$, i.e., for $n \in \mathbb{N}$,

$$\begin{array}{lrcl} \tau \text{ method:} & \mathrm{Spec}_\varepsilon A & \subset & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}} \\ \pi \text{ method:} & \mathrm{Spec}_\varepsilon A & \subset & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon'_n} A^{\mathrm{per}}_{n,k}} \\ \tau_1 \text{ method:} & \mathrm{Spec}_\varepsilon A & \subset & \Gamma^n_{\varepsilon+\varepsilon''_n}(A) = \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)}, \end{array}$$

with explicit and optimised formulae for $\varepsilon_n, \varepsilon'_n, \varepsilon''_n$. N.B. $\gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)$ can be interpreted as a pseudospectrum for a rectangular matrix.

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4. Sketched extension to A band-dominated, and reduction of $\bigcup_{k\in\mathbb{Z}}$ to a finite union, illustrating this by the **pseudoergodic** case.

J. Spectr. Theory 14 (2024), 719–804
DOI 10.4171/JST/514

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On spectral inclusion sets and computing the spectra and pseudospectra of bounded linear operators

Simon Chandler-Wilde, Ratchanikorn Chonchaiya, and Marko Lindner

Abstract. In this paper, we derive novel families of inclusion sets for the spectra and pseudospectra of large classes of bounded linear operators, and establish convergence of particular sequences of these inclusion sets to the spectrum or pseudospectrum, as appropriate. Our results apply, in particular, to bounded linear operators on a separable Hilbert space that, with respect to some orthonormal basis, have a representation as a bi-infinite matrix that is banded or band-dominated. More generally, our results apply in cases where the matrix entries themselves are bounded linear operators on some Banach space. In the scalar matrix entry case, we show that our methods, given the input information we assume, lead to a sequence of approximations to the spectrum, each element of which can be computed in finitely many arithmetic operations, so that, with our assumed inputs, the problem of determining the spectrum of a band-dominated operator has solvability complexity index one in the sense of Ben-Artzi et al. (2020). As a concrete and substantial application, we apply our methods to the determination of the spectra of non-self-adjoint bi-infinite tridiagonal matrices that are pseudoergodic in the sense of Davies [Commun. Math. Phys. 216 (2001), 687–704].

Dedicated to Prof. E. Brian Davies on the occasion of his 80th birthday

Extensions and future work

We've seen inclusion sets for the **spectrum** of **bi-infinite matrices**, i.e., operators on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}, X)$. These lead to results for **semi-infinite matrices** which lead to sequences of convergent inclusion sets also for the **essential spectrum**.

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We have inclusion set familes also for **finite matrices**, as a non-trivial (and often sharp) extension of block-matrix versions of **Gershgorin's theorem** – see arXiv:2408.03883

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Our bi-infinite matrix results depend on the group structure of $(\mathbb{Z}, +)$. In work with **Christian Seifert** (TU Hamburg) we replace $\mathbb{Z}$ with a general Abelian group G, so our matrices act on $\ell^2(G)$ or $\ell^2(G, X)$. E.g. $G = \mathbb{Z}^d$, $G =$ finite group, $\ldots$.

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Exciting applications in mathematical physics! E.g., project just started with Marko, Christian, **Matt Colbrook** (Cambridge), . . . on spectra of **almost-periodic operators** modelling **quasi-crystals**, cf. Hege, Moscolari, Teufel, *Phys. Rev. B* (2022).

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