Computing the Spectra and Pseudospectra of Band-Dominated and Random Operators

Simon Chandler-Wilde

March 2024, Heriot-Watt Seminar

University of Reading

...with the help of...

This talk is based on joint work, see
`https://arxiv.org/abs/2401.03984`, with

- Marko Lindner, TU Hamburg, Germany
- Ratchanikorn Chonchaiya, King Mongkut's University of Technology, Thailand

and supported by Marie Curie Grants of the European Union.

What is this talk about?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

What is this talk about?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

Answer. A qualified **yes**, if the matrix representation of A, with respect to some orthonormal sequence, is banded or band-dominated.

What is this talk about?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

Answer. A qualified **yes**, if the matrix representation of A, with respect to some orthonormal sequence, is banded or band-dominated.

Novelty? We know how to construct U_n satisfying (iii) with $U_n \to \mathrm{Spec}_\varepsilon A$, the ε-pseudospectrum, for band-dominated A (see Hansen 2011, Ben-Artzi, Colbrook, Hansen, Nevanlinna, Seidel 2015, 2020). But not known how to achieve (ii) and (iii)

What is this talk about?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

Answer. A qualified **yes**, if the matrix representation of A, with respect to some orthonormal sequence, is banded or band-dominated.

Novelty? We know how to construct U_n satisfying (iii) with $U_n \to \mathrm{Spec}_\varepsilon A$, the ε-pseudospectrum, for band-dominated A (see Hansen 2011, Ben-Artzi, Colbrook, Hansen, Nevanlinna, Seidel 2015, 2020). But not known how to achieve (ii) and (iii), and certainly not (i)-(iii).

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x,x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x,y) = \sum_{j\in\mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j\in\mathbb{Z}} |x_j|^2.$$

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x, x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x, y) = \sum_{j \in \mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j \in \mathbb{Z}} |x_j|^2.$$

If E, Y are Hilbert spaces, A is a **bounded linear operator** from E to Y, in symbols $A \in L(E, Y)$, if

$$A(\lambda x) = \lambda Ax, \quad A(x + y) = Ax + Ay, \quad \forall \lambda \in \mathbb{C}, \ x, y \in E,$$

and, for some $C \geq 0$,

$$\|Ax\| \leq C\|x\|, \quad \forall x \in E.$$

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x,y) and norm $\|x\| = \sqrt{(x,x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x,y) = \sum_{j\in\mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j\in\mathbb{Z}} |x_j|^2.$$

If E, Y are Hilbert spaces, A is a **bounded linear operator** from E to Y, in symbols $A \in L(E,Y)$, if

$$A(\lambda x) = \lambda\, Ax, \quad A(x+y) = Ax + Ay, \quad \forall \lambda \in \mathbb{C},\ x, y \in E,$$

and, for some $C \geq 0$,

$$\|Ax\| \leq C\|x\|, \quad \forall x \in E.$$

For $A \in L(E,Y)$ the **norm** and **lower norm** of A are

$$\|A\| := \sup_{x\in E\setminus\{0\}} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \nu(A) := \inf_{x\in E\setminus\{0\}} \frac{\|Ax\|}{\|x\|}.$$

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x, x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x, y) = \sum_{j \in \mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j \in \mathbb{Z}} |x_j|^2.$$

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x,x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x, y) = \sum_{j\in\mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j\in\mathbb{Z}} |x_j|^2.$$

If $A \in L(E, Y)$, the **adjoint** of A, denoted A^*, is the unique $A^* \in L(Y, E)$ satisfying

$$(Ax, y) = (x, A^*y), \quad x \in E, \ y \in Y.$$

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x, x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x, y) = \sum_{j\in\mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j\in\mathbb{Z}} |x_j|^2.$$

If $A \in L(E, Y)$, the **adjoint** of A, denoted A^*, is the unique $A^* \in L(Y, E)$ satisfying

$$(Ax, y) = (x, A^*y), \quad x \in E, \; y \in Y.$$

We call $A \in L(E) := L(E, E)$

- **self-adjoint** if $A^* = A$
- **normal** if $AA^* = A^*A$

Bounded linear operators between Hilbert spaces

$A \in L(E) := L(E, E)$ is said to be **invertible** if is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$.

Bounded linear operators between Hilbert spaces

$A \in L(E) := L(E, E)$ is said to be **invertible** if is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$. N.B.

$$A \text{ is not invertible} \quad \Leftrightarrow \quad \mu(A) := \min(\nu(A), \nu(A^*)) = 0,$$

$A \in L(E) := L(E, E)$ is said to be **invertible** if is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$. N.B.

$$A \text{ is not invertible} \quad \Leftrightarrow \quad \mu(A) := \min(\nu(A), \nu(A^*)) = 0,$$

and, if A is invertible, then

- A^* is invertible and $\|A^{-1}\| = \|(A^*)^{-1}\|$

Bounded linear operators between Hilbert spaces

$A \in L(E) := L(E, E)$ is said to be **invertible** if is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$. N.B.

$$A \text{ is not invertible} \quad \Leftrightarrow \quad \mu(A) := \min(\nu(A), \nu(A^*)) = 0,$$

and, if A is invertible, then

- A^* is invertible and $\|A^{-1}\| = \|(A^*)^{-1}\|$
- $\nu(A) = 1/\|A^{-1}\|$

Bounded linear operators between Hilbert spaces

$A \in L(E) := L(E, E)$ is said to be **invertible** if is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$. N.B.

$$A \text{ is not invertible} \quad \Leftrightarrow \quad \mu(A) := \min(\nu(A), \nu(A^*)) = 0,$$

and, if A is invertible, then

- A^* is invertible and $\|A^{-1}\| = \|(A^*)^{-1}\|$
- $\nu(A) = 1/\|A^{-1}\|$
- $\nu(A) = 1/\|A^{-1}\| = 1/\|(A^*)^{-1}\| = \nu(A^*) = \mu(A)$.

Bounded linear operators between Hilbert spaces

$A \in L(E) := L(E, E)$ is said to be **invertible** if is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$. N.B.

$$A \text{ is not invertible} \quad \Leftrightarrow \quad \mu(A) := \min(\nu(A), \nu(A^*)) = 0,$$

and, if A is invertible, then

- A^* is invertible and $\|A^{-1}\| = \|(A^*)^{-1}\|$
- $\nu(A) = 1/\|A^{-1}\|$
- $\nu(A) = 1/\|A^{-1}\| = 1/\|(A^*)^{-1}\| = \nu(A^*) = \mu(A)$.

With the conventions that $\|A^{-1}\| := \infty$ if A is not invertible and $1/\infty := 0$,

$$\mu(A) = 1/\|A^{-1}\|, \quad \text{for all } A \in L(Y).$$

For $A \in L(E)$ the **spectrum** of A is

$$\mathrm{Spec}\, A := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}.$$

N.B. this is just the set of eigenvalues if E is finite dimensional.

Spectrum and Pseudospectrum

For $A \in L(E)$ the **spectrum** of A is

$$\mathrm{Spec}\, A := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}.$$

For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\mathrm{Spec}_\varepsilon A \quad := \quad \{\lambda \in \mathbb{C} : 1/\|(A - \lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\}$$

Spectrum and Pseudospectrum

For $A \in L(E)$ the **spectrum** of A is

$$\mathrm{Spec}\, A := \{\lambda \in \mathbb{C} : A-\lambda I \text{ is not invertible}\} = \{\lambda \in \mathbb{C} : \mu(A-\lambda I) = 0\}.$$

For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned}\mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A-\lambda I)^{-1}\| \le \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A-\lambda I) \le \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,}\end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

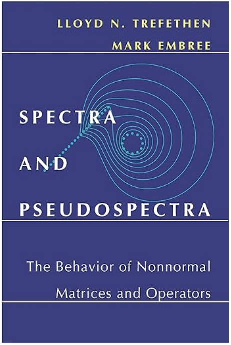
Spectrum and Pseudospectrum

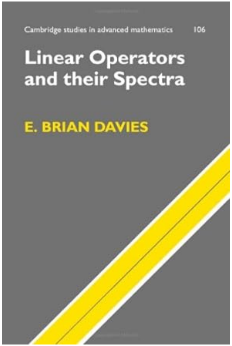
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A - \lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

More on pseudospectra: Trefethen & Embree 2005, Davies 2007





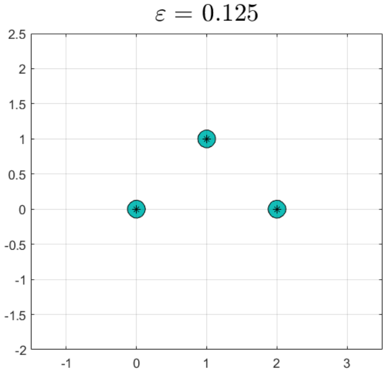
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned}\mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A-\lambda I)^{-1}\| \le \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A-\lambda I) \le \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,}\end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 1.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+\mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

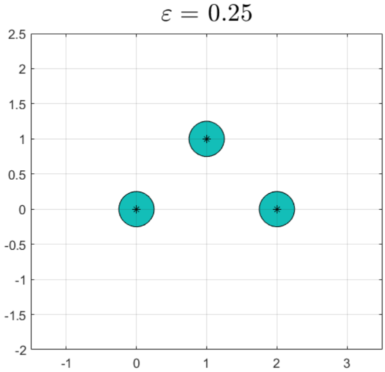
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A \quad &:= \quad \{\lambda \in \mathbb{C} : 1/\|(A - \lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\} \\ &\supset \quad \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 1.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 + \mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

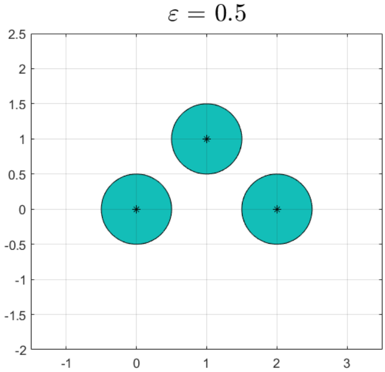
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A - \lambda I)^{-1}\| \le \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \le \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 1.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 + \mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

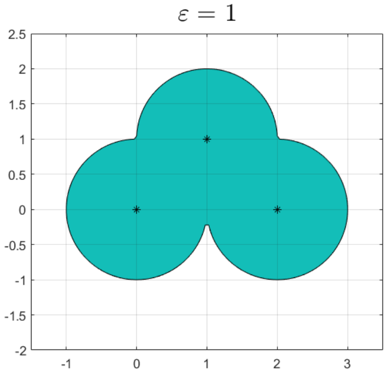
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned}\mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A-\lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A-\lambda I) \leq \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,}\end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 1.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+\mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

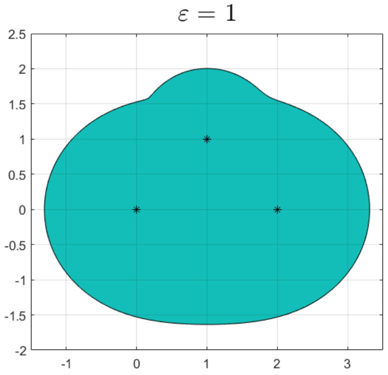
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A-\lambda I)^{-1}\| \le \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A-\lambda I) \le \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 2.

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{15} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{1}{6} + \frac{\mathrm{i}}{10} \\ 0 & 0 & 1 + \mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

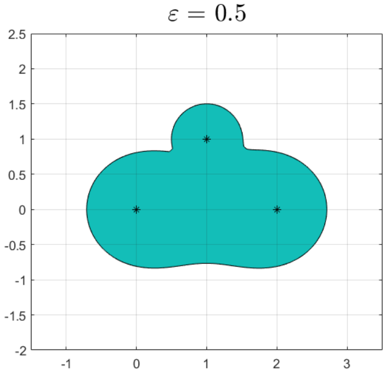
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A-\lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A-\lambda I) \leq \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 2.

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{15} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{1}{6} + \frac{\mathrm{i}}{10} \\ 0 & 0 & 1+\mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

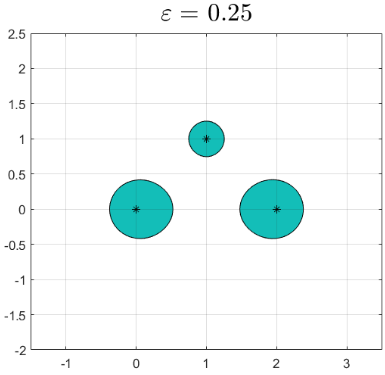
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A &:= \{\lambda \in \mathbb{C} : 1/\|(A - \lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\} \\ &\supset \mathrm{Spec}\, A + \varepsilon\overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 2.

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{15} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{1}{6} + \frac{\mathrm{i}}{10} \\ 0 & 0 & 1 + \mathrm{i} \end{bmatrix}$$



Spectrum and Pseudospectrum

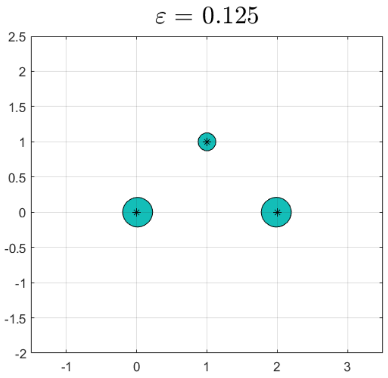
For $A \in L(E)$ and $\varepsilon > 0$ the (closed) ε-**pseudospectrum** of A is

$$\begin{aligned} \mathrm{Spec}_\varepsilon A \; &:= \; \{\lambda \in \mathbb{C} : 1/\|(A-\lambda I)^{-1}\| \leq \varepsilon\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) \leq \varepsilon\} \\ &\supset \; \mathrm{Spec}\, A + \varepsilon \overline{\mathbb{D}}, \text{ with equality if } A \text{ is normal,} \end{aligned}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 2.

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} & -\frac{2}{15} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{1}{6} + \frac{\mathrm{i}}{10} \\ 0 & 0 & 1 + \mathrm{i} \end{bmatrix}$$



Hausdorff convergence of sets

Let

- $\mathbb{C}^B$:= set of non-empty **bounded** subsets of $\mathbb{C}$
- $\mathbb{C}^C$:= set of non-empty **compact** subsets of $\mathbb{C}$

Hausdorff convergence of sets

Let

- $\mathbb{C}^B$:= set of non-empty **bounded** subsets of $\mathbb{C}$
- $\mathbb{C}^C$:= set of non-empty **compact** subsets of $\mathbb{C}$

For $S, T \in \mathbb{C}^B$ let

$$d(S,T) := \inf\{\varepsilon \geq 0 : S \subset T + \varepsilon\mathbb{D} \text{ and } T \subset S + \varepsilon\mathbb{D}\}.$$

N.B. $d(\cdot,\cdot)$ is the **Hausdorff metric** on $\mathbb{C}^C$.

Hausdorff convergence of sets

Let

- $\mathbb{C}^B$:= set of non-empty **bounded** subsets of $\mathbb{C}$
- $\mathbb{C}^C$:= set of non-empty **compact** subsets of $\mathbb{C}$

For $S, T \in \mathbb{C}^B$ let

$$d(S,T) := \inf\{\varepsilon \ge 0 : S \subset T + \varepsilon\mathbb{D} \text{ and } T \subset S + \varepsilon\mathbb{D}\}.$$

N.B. $d(\cdot,\cdot)$ is the **Hausdorff metric** on $\mathbb{C}^C$. For $(S_n) \subset \mathbb{C}^B$ and $S \in \mathbb{C}^B$ write $S_n \to S$ if $d(S_n, S) \to 0$ as $n \to \infty$.

Hausdorff convergence of sets

Let

- $\mathbb{C}^B$:= set of non-empty **bounded** subsets of $\mathbb{C}$
- $\mathbb{C}^C$:= set of non-empty **compact** subsets of $\mathbb{C}$

For $S, T \in \mathbb{C}^B$ let

$$d(S,T) := \inf \{\varepsilon \geq 0 : S \subset T + \varepsilon\mathbb{D} \text{ and } T \subset S + \varepsilon\mathbb{D}\}.$$

N.B. $d(\cdot,\cdot)$ is the **Hausdorff metric** on $\mathbb{C}^C$. For $(S_n) \subset \mathbb{C}^B$ and $S \in \mathbb{C}^B$ write $S_n \to S$ if $d(S_n, S) \to 0$ as $n \to \infty$.

Lemma. If $(S_n) \subset \mathbb{C}^C$ and $S_1 \supset S_2 \supset \ldots$, then $S_n \to S_\infty := \bigcap_{n\in\mathbb{N}} S_n$ as $n \to \infty$.

Hausdorff convergence of sets

Let

- $\mathbb{C}^B$:= set of non-empty **bounded** subsets of $\mathbb{C}$
- $\mathbb{C}^C$:= set of non-empty **compact** subsets of $\mathbb{C}$

For $S, T \in \mathbb{C}^B$ let

$$d(S,T) := \inf\{\varepsilon \geq 0 : S \subset T + \varepsilon\mathbb{D} \text{ and } T \subset S + \varepsilon\mathbb{D}\}.$$

N.B. $d(\cdot,\cdot)$ is the **Hausdorff metric** on $\mathbb{C}^C$. For $(S_n) \subset \mathbb{C}^B$ and $S \in \mathbb{C}^B$ write $S_n \to S$ if $d(S_n, S) \to 0$ as $n \to \infty$.

Lemma. If $(S_n) \subset \mathbb{C}^C$ and $S_1 \supset S_2 \supset \ldots$, then $S_n \to S_\infty := \bigcap_{n\in\mathbb{N}} S_n$ as $n \to \infty$.

Corollary. If $\varepsilon_1 > \varepsilon_2 > \ldots > 0$, in which case $\varepsilon_n \to \varepsilon \geq 0$ as $n \to \infty$, then

$$\mathrm{Spec}_{\varepsilon_n} A \to \mathrm{Spec}_{\varepsilon} A \qquad \text{N.B.} \quad \mathrm{Spec}_0 A := \mathrm{Spec}\, A.$$

Matrix representation of A

Suppose $(e_j)_{j\in\mathbb{Z}}$ is an orthonormal basis for a separable Hilbert space E and $A \in L(E)$. Then the **matrix representation** of A is $[A] = [a_{ij}]_{i,j\in\mathbb{Z}}$, where

$$a_{ij} = (Ae_j, e_i), \qquad i, j \in \mathbb{Z},$$

and $\mathrm{Spec}\, A = \mathrm{Spec}\, [A]$, $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon [A]$, $\varepsilon > 0$, where $[A] \in L(\ell^2)$ is defined by

$$([A]x)_i = \sum_{j\in\mathbb{Z}} a_{ij} x_j, \quad i \in \mathbb{Z}.$$

Matrix representation of A

Suppose $(e_j)_{j\in\mathbb{Z}}$ is an orthonormal basis for a separable Hilbert space E and $A \in L(E)$. Then the **matrix representation** of A is $[A] = [a_{ij}]_{i,j\in\mathbb{Z}}$, where

$$a_{ij} = (Ae_j, e_i), \qquad i, j \in \mathbb{Z},$$

and $\mathrm{Spec}\, A = \mathrm{Spec}\, [A]$, $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon [A]$, $\varepsilon > 0$, where $[A] \in L(\ell^2)$ is defined by

$$([A]x)_i = \sum_{j\in\mathbb{Z}} a_{ij}x_j, \quad i \in \mathbb{Z}.$$

The above makes clear we can assume $E = \ell^2 = \ell^2(\mathbb{Z})$, in which case we will abbreviate $[A]$ as A.

Matrix representation of A

Suppose $(e_j)_{j\in\mathbb{Z}}$ is an orthonormal basis for a separable Hilbert space E and $A \in L(E)$. Then the **matrix representation** of A is $[A] = [a_{ij}]_{i,j\in\mathbb{Z}}$, where

$$a_{ij} = (Ae_j, e_i), \qquad i,j \in \mathbb{Z},$$

and $\mathrm{Spec}\,A = \mathrm{Spec}\,[A]$, $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon [A]$, $\varepsilon > 0$, where $[A] \in L(\ell^2)$ is defined by

$$([A]x)_i = \sum_{j\in\mathbb{Z}} a_{ij}x_j, \quad i \in \mathbb{Z}.$$

The above makes clear we can assume $E = \ell^2 = \ell^2(\mathbb{Z})$, in which case we will abbreviate $[A]$ as A.

We will say that $[A]$ is **banded** with **bandwidth** $w \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ if $a_{ij} = 0$ for $|i - j| > w$.

Suppose $(e_j)_{j\in\mathbb{Z}}$ is an orthonormal basis for a separable Hilbert space E and $A \in L(E)$. Then the **matrix representation** of A is $[A] = [a_{ij}]_{i,j\in\mathbb{Z}}$, where

$$a_{ij} = (Ae_j, e_i), \qquad i, j \in \mathbb{Z},$$

and $\mathrm{Spec}\, A = \mathrm{Spec}\, [A]$, $\mathrm{Spec}_\varepsilon A = \mathrm{Spec}_\varepsilon [A]$, $\varepsilon > 0$, where $[A] \in L(\ell^2)$ is defined by

$$([A]x)_i = \sum_{j\in\mathbb{Z}} a_{ij} x_j, \quad i \in \mathbb{Z}.$$

The above makes clear we can assume $E = \ell^2 = \ell^2(\mathbb{Z})$, in which case we will abbreviate $[A]$ as A.

We will say that $[A]$ is **banded** with **bandwidth** $w \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ if $a_{ij} = 0$ for $|i - j| > w$.

We will say that $[A]$ is **band-dominated** if there exists a sequence $(A_n) \subset L(E)$ such that each $[A_n]$ is banded and $\|A - A_n\| \to 0$ as $n \to \infty$.

The tridiagonal case

Let's consider first bi-infinite matrices of the form

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_2 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $\alpha = (\alpha_i),\ \beta = (\beta_i)$ and $\gamma = (\gamma_i)$ are bounded sequences of complex numbers.

Inclusion sets for $\mathrm{Spec}_\varepsilon A$, $\varepsilon \geq 0$.

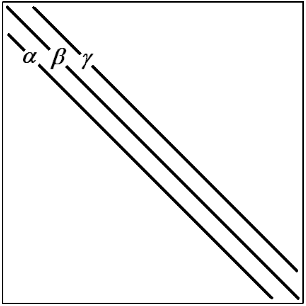
$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \beta_{-2} & \gamma_{-1} & & & & \\ & \alpha_{-2} & \beta_{-1} & \gamma_0 & & & \\ & & \alpha_{-1} & \beta_0 & \gamma_1 & & \\ & & & \alpha_0 & \beta_1 & \gamma_2 & \\ & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

Task

Compute **inclusion sets for spectrum and pseudospectra** of $A \in L(\ell^2) = L(\ell^2(\mathbb{Z}))$.

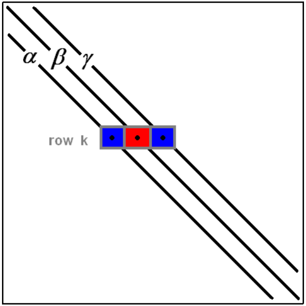
Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:



Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:

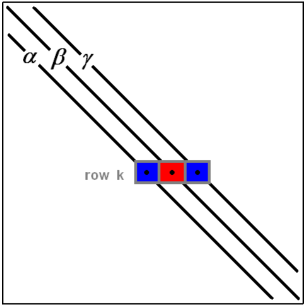


For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:

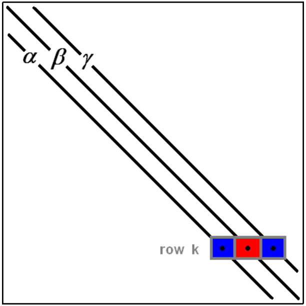


For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:

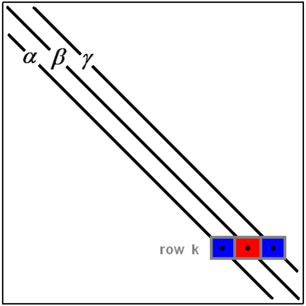


For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:



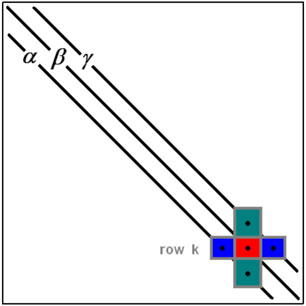
For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

$$\operatorname{Spec} A \subset \overline{\bigcup_{k\in\mathbb{Z}} (a_{k,k} + (\|\alpha\|_\infty + \|\gamma\|_\infty)\mathbb{D})}.$$

Inspiration: Gershgorin discs

Here is our tridiagonal bi-infinite matrix:



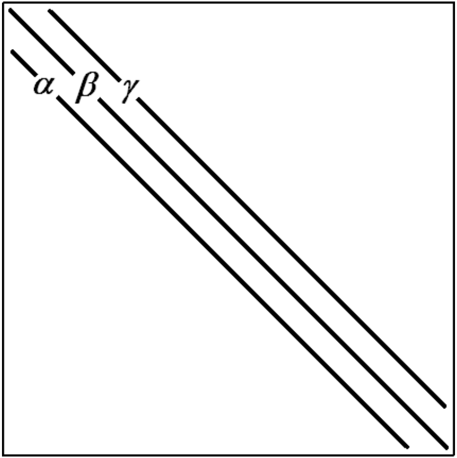
For every row k, consider the **Gershgorin disc** with

center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

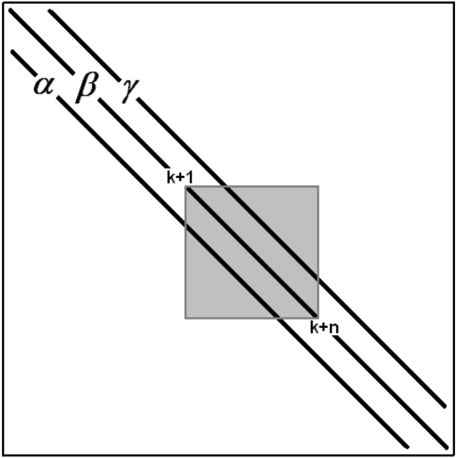
$$\mathrm{Spec}\, A \subset \overline{\bigcup_{k\in\mathbb{Z}} (a_{k,k} + (\|\alpha\|_\infty + \|\gamma\|_\infty)\mathbb{D})}.$$

Our new strategy

Look at (pseudo)spectra of the **finite principal submatrices** of A:

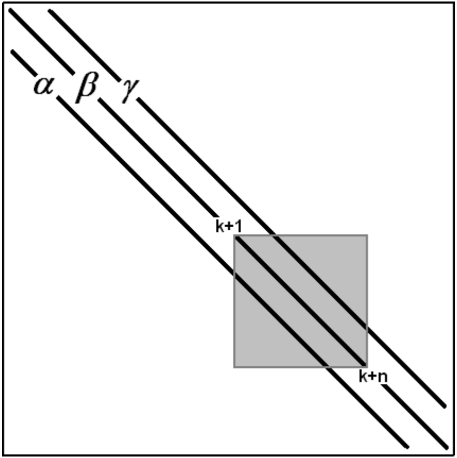


Look at (pseudo)spectra of the **finite principal submatrices** of A:



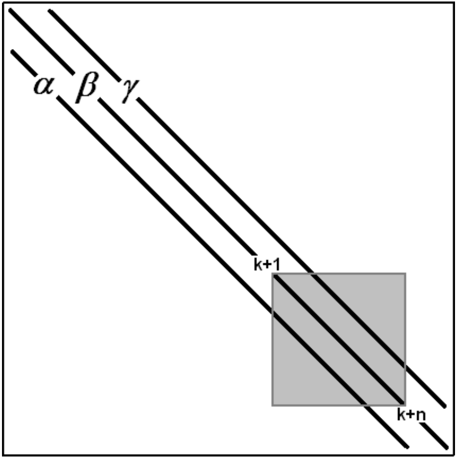
Our new strategy

Look at (pseudo)spectra of the **finite principal submatrices** of A:



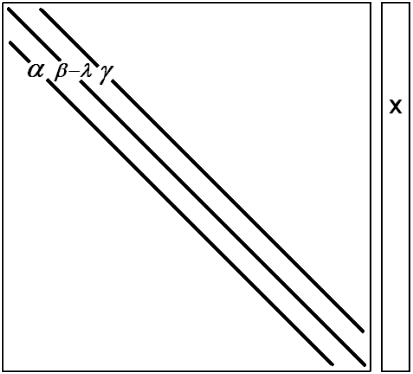
Our new strategy

Look at (pseudo)spectra of the **finite principal submatrices** of A:



τ method: finite principal submatrices

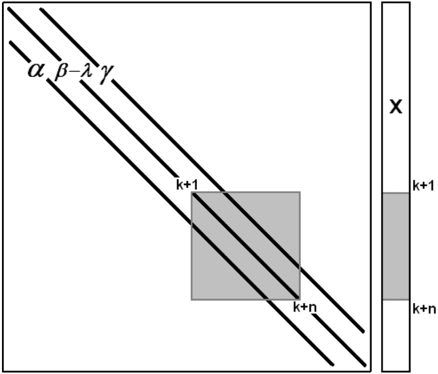
Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;\le\; \varepsilon\,\|x\|$$

τ method: finite principal submatrices

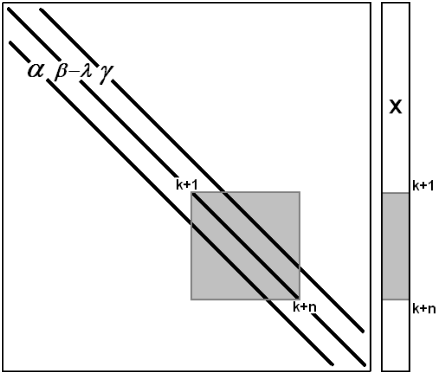
Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \;\leq\; \varepsilon\, \|x\|$$

τ method: finite principal submatrices

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



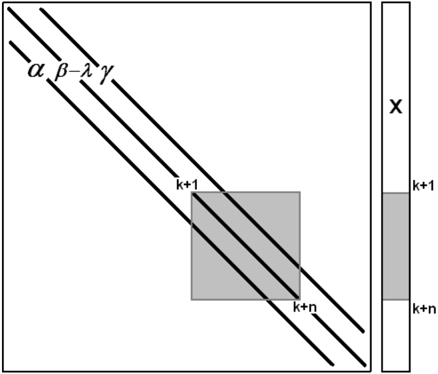
$$\|(A - \lambda I)\, x\| \;\leq\; \varepsilon\, \|x\|$$

Claim: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

τ method: finite principal submatrices

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



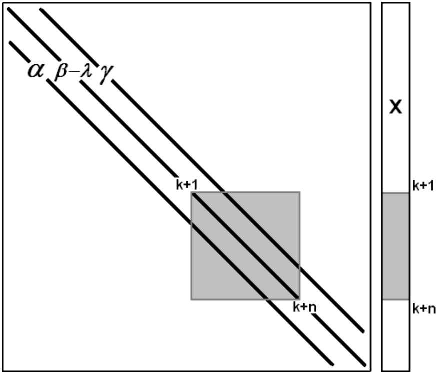
$$\|(A - \lambda I)\, x\| \;\leq\; \varepsilon \, \|x\|$$

Claim: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\Leftarrow \sum_k \|(A_{n,k} - \lambda I_n)\, x_{n,k}\|^2 \leq (\varepsilon + \varepsilon_n)^2 \sum_k \|x_{n,k}\|^2$$

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.

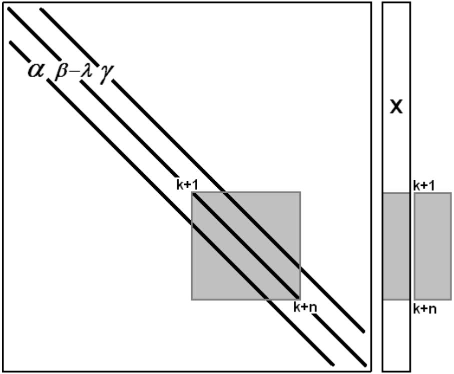


$$\|(A - \lambda I)\,x\| \;\le\; \varepsilon\,\|x\|$$

Fact: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| \le (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;\leq\; \varepsilon\,\|x\|$$

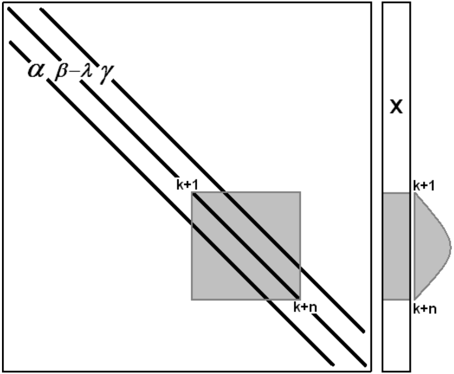
Fact: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\varepsilon_n \;=\; \frac{1}{\sqrt{n}}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

τ method: finite principal submatrices

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;\le\; \varepsilon\,\|x\|$$

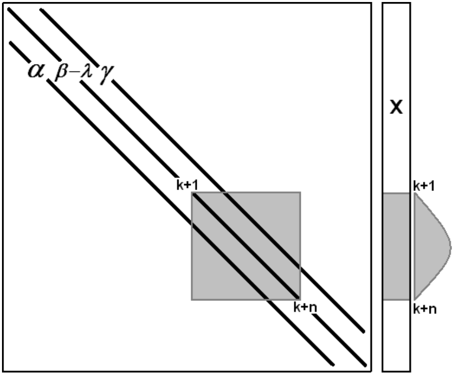
Fact: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| \le (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\varepsilon_n \le 2\sin\frac{\pi}{2(n+2)}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

τ method: finite principal submatrices

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;\leq\; \varepsilon\,\|x\|$$

Fact: $\exists k \in \mathbb{Z}$:

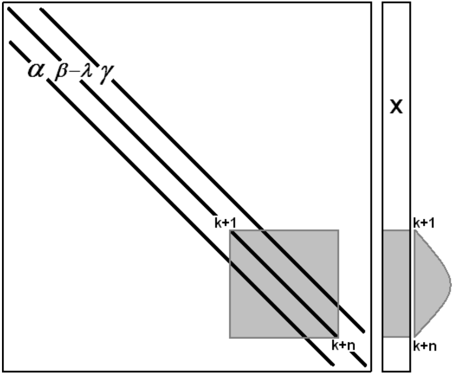
$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\varepsilon_n \leq 2\sin\frac{\pi}{2(n+2)}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow \quad \lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}$$

τ method: finite principal submatrices

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \;\leq\; \varepsilon\, \|x\|$$

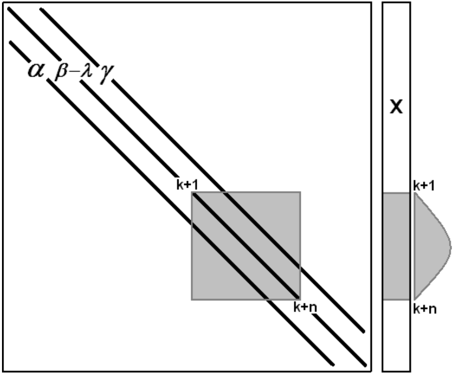
Fact: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\varepsilon_n \leq 2 \sin \frac{\pi}{2(n+2)} (\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow \;\; \lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}$$

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\,x\| \;\leq\; \varepsilon\,\|x\|$$

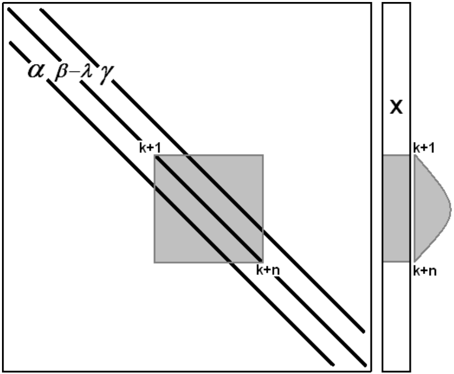
Fact: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\,x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\,\|x_{n,k}\|$$

$$\varepsilon_n \leq 2\sin\frac{\pi}{2(n+2)}(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow \quad \lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}$$

Let $\lambda \in \mathrm{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)\, x\| \;\leq\; \varepsilon\, \|x\|$$

Fact: $\exists k \in \mathbb{Z}$:

$$\|(A_{n,k} - \lambda I_n)\, x_{n,k}\| \leq (\varepsilon + \varepsilon_n)\, \|x_{n,k}\|$$

$$\varepsilon_n \leq 2 \sin \frac{\pi}{2(n+2)} (\|\alpha\|_\infty + \|\gamma\|_\infty)$$

$$\Rightarrow \;\; \lambda \in \mathrm{Spec}_{\varepsilon + \varepsilon_n} A_{n,k}$$

τ method: finite principal submatrices

So we get

Inclusion Set

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}, \quad \varepsilon \geq 0,$$

where

$$\varepsilon_n \;\leq\; 2\sin\left(\frac{\pi}{2(n+2)}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty),$$

so $\varepsilon_n = O(n^{-1})$ as $n \to \infty$.

τ method: finite principal submatrices

So we get

Inclusion Set

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}, \quad \varepsilon \geq 0,$$

where

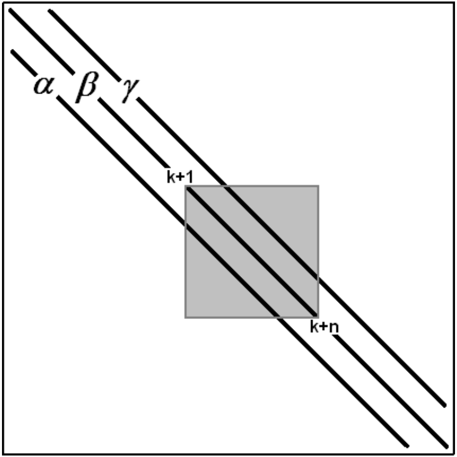
$$\varepsilon_n \;\leq\; 2\sin\left(\frac{\pi}{2(n+2)}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty),$$

so $\varepsilon_n = O(n^{-1})$ as $n \to \infty$. Putting $n = 1$ and $\varepsilon = 0$ we recover Gershgorin:

$$\mathrm{Spec}\, A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_1} A_{1,k}} = \overline{\bigcup_{k\in\mathbb{Z}} (a_{k,k} + (\|\alpha\|_\infty + \|\gamma\|_\infty)\mathbb{D})}.$$

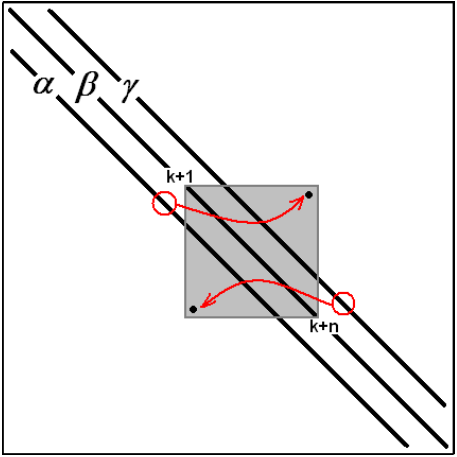
π method: **periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised" (cf. Colbrook 2020, which uses **single large periodised finite section**)



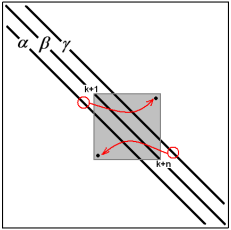
π method: **periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised" (cf. Colbrook 2020, which uses **single large periodised finite section**)



π method: **periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are "periodised" (cf. Colbrook 2020, which uses **single large periodised finite section**)



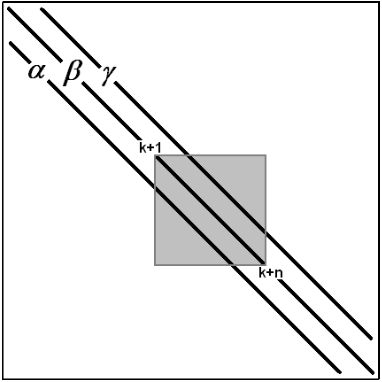
very similar computations show that

$$\operatorname{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \operatorname{Spec}_{\varepsilon+\varepsilon_n'} A^{\mathrm{per}}_{n,k}}, \quad \varepsilon \geq 0,$$

$$\text{with} \quad \varepsilon_n' = 2\sin\left(\frac{\pi}{2n}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty).$$

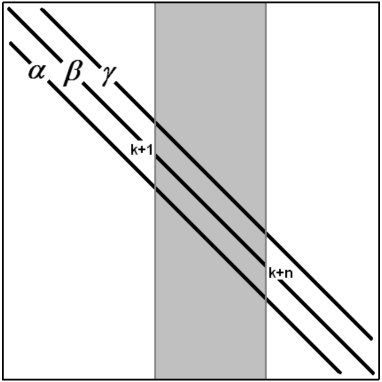
Here is another idea: τ_1 method

Instead of



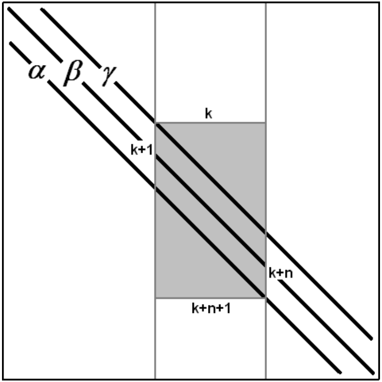
Here is another idea: τ_1 method

We do a **“one-sided”** truncation.



Here is another idea: τ_1 method

We do a **"one-sided"** truncation.



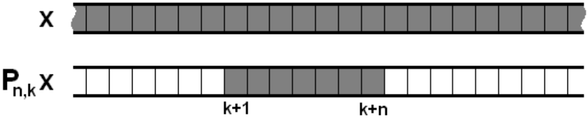
I.e., we work with **rectangular** finite submatrices.

This is motivated by work of Davies 1998, Davies & Plum 2004, and Hansen 2008, 2011, in which A is approximated by a **single large rectangular finite section**.

τ_1 method: projection operator

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $P_{n,k} : \ell^2 \to \ell^2$ denote the projection

$$(P_{n,k}x)(i) := \begin{cases} x(i), & i \in \{k+1, ..., k+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

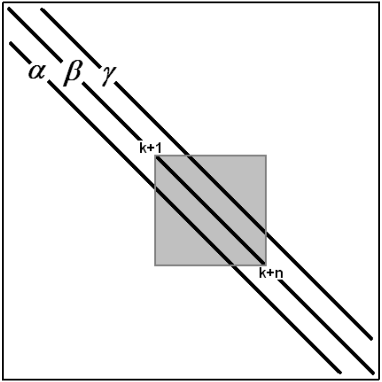


Further, we put

$$E_{n,k} := \operatorname{im} P_{n,k}.$$

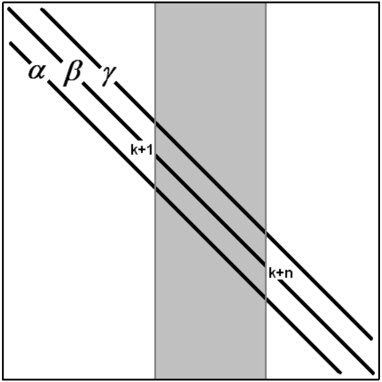
τ_1 method: truncations

τ method:



$$P_{n,k}(A-\lambda I)|_{E_{n,k}}$$

τ_1 method:



$$(A-\lambda I)|_{E_{n,k}}$$

τ_1 method: τ method revisited

τ method:

$$\lambda \in \mathrm{Spec}_\varepsilon A \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k} A|_{E_{n,k}}\right)$$

τ_1 method: τ method revisited

τ method:

$\lambda \in \mathrm{Spec}_\varepsilon A \quad \Longrightarrow \quad$ For some $k \in \mathbb{Z}$:

$$\lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}A|_{E_{n,k}}\right)$$

i.e. $\mu\left(P_{n,k}(A-\lambda I)|_{E_{n,k}}\right) \leq \varepsilon + \varepsilon_n$

τ_1 method: τ method revisited

τ **method:**

$$\lambda \in \mathrm{Spec}_\varepsilon A \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}A|_{E_{n,k}}\right)$$

$$\text{i.e.} \quad \mu\left(P_{n,k}(A-\lambda I)|_{E_{n,k}}\right) \leq \varepsilon+\varepsilon_n$$

$$\text{i.e.} \quad \nu\left(P_{n,k}(A-\lambda I)|_{E_{n,k}}\right) \leq \varepsilon+\varepsilon_n \text{ or}$$

$$\nu\left(P_{n,k}(A-\lambda I)^*|_{E_{n,k}}\right) \leq \varepsilon+\varepsilon_n$$

τ_1 method: τ method revisited

τ method:

$$\lambda \in \mathrm{Spec}_\varepsilon A \quad \Longrightarrow \quad \text{For some } k \in \mathbb{Z}:$$

$$\lambda \in \mathrm{Spec}_{\varepsilon+\varepsilon_n}\left(P_{n,k}A|_{E_{n,k}}\right)$$

$$\text{i.e.} \quad \mu\left(P_{n,k}(A-\lambda I)|_{E_{n,k}}\right) \le \varepsilon + \varepsilon_n$$

$$\text{i.e.} \quad \nu\left(P_{n,k}(A-\lambda I)|_{E_{n,k}}\right) \le \varepsilon + \varepsilon_n \text{ or}$$

$$\nu\left(P_{n,k}(A-\lambda I)^*|_{E_{n,k}}\right) \le \varepsilon + \varepsilon_n$$

τ_1 idea is just **drop the $P_{n,k}$'s.**

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \leq \varepsilon.$$

(Analogue of $\mathrm{Spec}_\varepsilon A_{n,k}$ in the τ method.)

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right),\, \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \;\leq\; \varepsilon.$$

(Analogue of $\mathrm{Spec}_\varepsilon A_{n,k}$ in the τ method.) Put

$$\Gamma_\varepsilon^n(A) \;:=\; \overline{\bigcup_{k\in\mathbb{Z}} \gamma_\varepsilon^{n,k}(A)}.$$

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \leq \varepsilon.$$

(Analogue of $\mathrm{Spec}_\varepsilon A_{n,k}$ in the τ method.) Put

$$\Gamma_\varepsilon^n(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma_\varepsilon^{n,k}(A)}.$$

Then (similarly to the τ and π-method inclusions)

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A),$$

with $$\varepsilon''_n = 2\sin\left(\frac{\pi}{2n+2}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \leq \varepsilon.$$

(Analogue of $\mathrm{Spec}_\varepsilon A_{n,k}$ in the τ method.) Put

$$\Gamma_\varepsilon^n(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma_\varepsilon^{n,k}(A)}.$$

Then (similarly to the τ and π-method inclusions)

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A),$$

$$\text{with} \qquad \varepsilon''_n = 2\sin\left(\frac{\pi}{2n+2}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

But now we also have that if $\lambda \in \gamma_\varepsilon^{n,k}(A)$, for some $k \in \mathbb{Z}$, then $\nu(A-\lambda) \leq \nu((A-\lambda I)|_{E_{n,k}}) \leq \varepsilon$ or $\nu((A-\lambda)^*) \leq \varepsilon$

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \leq \varepsilon.$$

(Analogue of $\mathrm{Spec}_\varepsilon A_{n,k}$ in the τ method.) Put

$$\Gamma_\varepsilon^n(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma_\varepsilon^{n,k}(A)}.$$

Then (similarly to the τ and π-method inclusions)

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A),$$

$$\text{with} \qquad \varepsilon''_n = 2\sin\left(\frac{\pi}{2n+2}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty)$$

But now we also have that if $\lambda \in \gamma_\varepsilon^{n,k}(A)$, for some $k \in \mathbb{Z}$, then $\nu(A-\lambda) \leq \nu((A-\lambda I)|_{E_{n,k}}) \leq \varepsilon$ or $\nu((A-\lambda)^*) \leq \varepsilon$, so $\mu(A-\lambda I) \leq \varepsilon$ and $\lambda \in \mathrm{Spec}_\varepsilon A$, so

$$\Gamma_\varepsilon^n(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A.$$

τ_1-method: spectral bounds

From the lower and upper bound

$$\Gamma_\varepsilon^n(A) \subset \mathrm{Spec}_\varepsilon A \quad \text{and} \quad \mathrm{Spec}_\varepsilon A \subset \Gamma_{\varepsilon+\varepsilon_n''}^n(A)$$

we get

Sandwich 1

$$\Gamma_\varepsilon^n(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma_{\varepsilon+\varepsilon_n''}^n(A), \quad \varepsilon \geq 0.$$

τ_1-method: spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{Spec}_\varepsilon A \quad \text{and} \quad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A), \quad \varepsilon \geq 0.$$

Sandwich 2

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \subset \quad \mathrm{Spec}_{\varepsilon+\varepsilon''_n} A, \quad \varepsilon \geq 0.$$

τ_1-method: spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{Spec}_\varepsilon A \quad \text{and} \quad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A), \quad \varepsilon \geq 0.$$

Sandwich 2

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \subset \quad \mathrm{Spec}_{\varepsilon+\varepsilon''_n} A, \quad \varepsilon \geq 0.$$

In particular, it follows, since $\mathrm{Spec}_{\varepsilon+\varepsilon''_n} A \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, that

$$\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \to \quad \mathrm{Spec}_\varepsilon A$$

τ_1-method: spectral bounds

From the lower and upper bound

$$\Gamma^n_\varepsilon(A) \subset \mathrm{Spec}_\varepsilon A \quad \text{and} \quad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A)$$

we get

Sandwich 1

$$\Gamma^n_\varepsilon(A) \quad \subset \quad \mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A), \quad \varepsilon \geq 0.$$

Sandwich 2

$$\mathrm{Spec}_\varepsilon A \quad \subset \quad \Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \subset \quad \mathrm{Spec}_{\varepsilon+\varepsilon''_n} A, \quad \varepsilon \geq 0.$$

In particular, it follows, since $\mathrm{Spec}_{\varepsilon+\varepsilon''_n} A \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, that

$$\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \quad \to \quad \mathrm{Spec}_\varepsilon A, \quad \text{in particular} \quad \Gamma^n_{\varepsilon''_n}(A) \quad \to \quad \mathrm{Spec}\, A.$$

The shift operator

Let's compute the τ, π, and τ_1 inclusion sets for $\mathrm{Spec}\, A$, i.e.

$$\begin{array}{ll} \tau \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n} A_{n,k}} \\ \pi \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon'_n} A^{\mathrm{per}}_{n,k}} \\ \tau_1 \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}, \end{array}$$

where

$$\gamma^{n,k}_{\varepsilon''_n}(A) = \left\{\lambda \in \mathbb{C} : \min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \le \varepsilon''_n\right\},$$

in the case that A is the **shift operator**, so that
$\alpha = (\ldots, 0, 0, \ldots)$, $\beta = (\ldots, 0, 0, \ldots)$, $\gamma = (\ldots, 1, 1, \ldots)$,

$$\mathrm{Spec}\, A = \mathbb{T} = \{z : |z| = 1\}$$

Let's compute the τ, π, and τ_1 inclusion sets for $\mathrm{Spec}\, A$, i.e.

$$\begin{array}{ll} \tau \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n} A_{n,k}} \\ \pi \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n'} A^{\mathrm{per}}_{n,k}} \\ \tau_1 \text{ method:} & \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon_n''}(A)}, \end{array}$$

where

$$\gamma^{n,k}_{\varepsilon_n''}(A) = \left\{\lambda \in \mathbb{C} : \min\left(\nu\left((A-\lambda I)|_{E_{n,k}}\right), \nu\left((A-\lambda I)^*|_{E_{n,k}}\right)\right) \le \varepsilon_n''\right\},$$

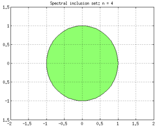
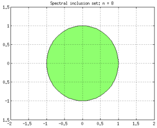
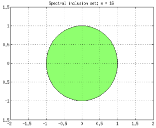
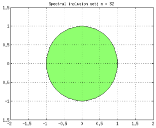
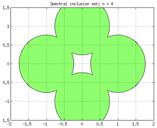
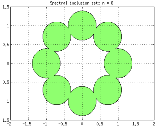
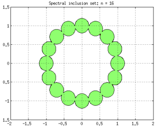
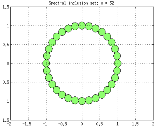
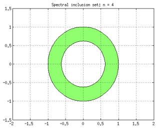
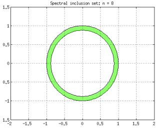
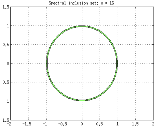
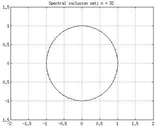
in the case that A is the **shift operator**, so that $\alpha = (\ldots, 0, 0, \ldots)$, $\beta = (\ldots, 0, 0, \ldots)$, $\gamma = (\ldots, 1, 1, \ldots)$,

$$\mathrm{Spec}\, A = \mathbb{T} = \{z : |z| = 1\},$$

$$\varepsilon_n, \varepsilon_n', \varepsilon_n'' \le 2\sin\left(\frac{\pi}{2n}\right)(\|\alpha\|_\infty + \|\gamma\|_\infty) = 2\sin\left(\frac{\pi}{2n}\right),$$

and the matrices $A_{n,k}$, $k \in \mathbb{Z}$, are all the same!

The shift operator

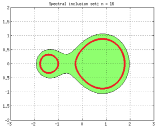
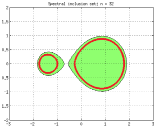
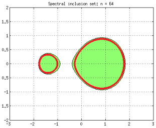
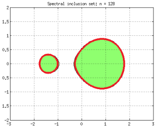
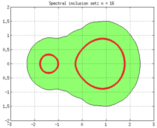
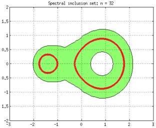
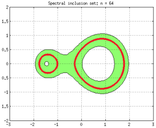
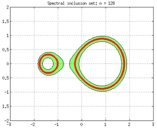
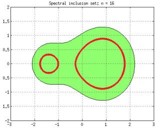
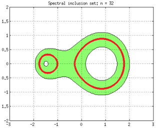
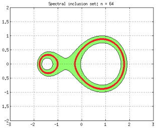
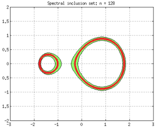
	$n = 4$	$n = 8$	$n = 16$	$n = 32$
τ method				
π method				
τ_1 method				

τ, π, and τ_1 methods: second example

We now look at a tridiagonal matrix A with 3-periodic diagonals:

1st sub-diagonal $\alpha = (\cdots, 0, 0, 0, \cdots)$
main diagonal $\beta = (\cdots, -\frac{3}{2}, 1, 1, \cdots)$
super-diagonal $\gamma = (\cdots, 1, 2, 1, \cdots)$

3-periodic example

	$n = 16$	$n = 32$	$n = 64$	$n = 128$
τ method				
π method				
τ_1 method				

Let's take stock: what were we trying to do?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

My claimed answer. A qualified **yes**, if the matrix representation of A, with respect to some orthonormal sequence, is banded or band-dominated.

Let's take stock: what were we trying to do?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

My claimed answer. A qualified **yes**, if the matrix representation of A, with respect to some orthonormal sequence, is banded or band-dominated.

If we put

$$U_n = \Gamma^n_{\varepsilon+\varepsilon''_n}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}$$

then (i) and (ii) are true, but only for **tridiagonal** A, and surely (iii) is not true?

Let's take stock: what were we trying to do?

Question. Given a bounded linear operator A on a Hilbert space E, can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\mathrm{Spec}\, A \subset U_n$ for each n;
- (ii) $U_n \to \mathrm{Spec}\, A$ as $n \to \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

My claimed answer. A qualified **yes**, if the matrix representation of A, with respect to some orthonormal sequence, is banded or band-dominated.

If we put

$$U_n = \Gamma^n_{\varepsilon+\varepsilon''_n}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}$$

then (i) and (ii) are true, but only for **tridiagonal** A, and surely (iii) is not true? What are the missing ingredients?

$$\text{If} \qquad U_n = \Gamma^n_{\varepsilon''_n}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}$$

then $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$, but only for **tridiagonal** A, and U_n can't be computed in finitely many operations.

Missing Ingredients (cf. Ben-Artzi et al. 2020)

The missing ingredients

$$\text{If} \qquad U_n = \Gamma^n_{\varepsilon''_n}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}$$

then $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$, but only for **tridiagonal** A, and U_n can't be computed in finitely many operations.

Missing Ingredients (cf. Ben-Artzi et al. 2020)

- Realize that the entries of the tridiagonal matrix can themselves be square matrices - extends to A **banded**.

The missing ingredients

$$\text{If} \qquad U_n = \Gamma^n_{\varepsilon''_n}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}$$

then $\mathrm{Spec}\,A \subset U_n$ and $U_n \to \mathrm{Spec}\,A$, but only for **tridiagonal** A, and U_n can't be computed in finitely many operations.

Missing Ingredients (cf. Ben-Artzi et al. 2020)

- Realize that the entries of the tridiagonal matrix can themselves be square matrices - extends to A **banded**.
- Perturbation argument extends to A **band-dominated**, approximated by A_n (banded), with $\delta_n := \|A - A_n\| \to 0$.

The missing ingredients

$$\text{If} \qquad U_n = \Gamma^n_{\varepsilon_n''}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon_n''}(A)}$$

then $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$, but only for **tridiagonal** A, and U_n can't be computed in finitely many operations.

Missing Ingredients (cf. Ben-Artzi et al. 2020)

- Realize that the entries of the tridiagonal matrix can themselves be square matrices - extends to A **banded**.
- Perturbation argument extends to A **band-dominated**, approximated by A_n (banded), with $\delta_n := \|A - A_n\| \to 0$.
- For τ method approximate $\overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n} A_{n,k}}$ by finite union $\bigcup_{k\in K_n^{\mathrm{fin}}} B_{n,k}$ where $\{B_{n,k} : k \in K_n^{\mathrm{fin}}\}$ is an ε-net (with $\varepsilon = 1/n$) for the compact set $\overline{\{A_{n,k} : k \in \mathbb{Z}\}}$. Similarly for τ_1 method.

The missing ingredients

$$\text{If} \qquad U_n = \Gamma^n_{\varepsilon''_n}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon''_n}(A)}$$

then $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$, but only for **tridiagonal** A, and U_n can't be computed in finitely many operations.

Missing Ingredients (cf. Ben-Artzi et al. 2020)

- Realize that the entries of the tridiagonal matrix can themselves be square matrices - extends to A **banded**.
- Perturbation argument extends to A **band-dominated**, approximated by A_n (banded), with $\delta_n := \|A - A_n\| \to 0$.
- For τ method approximate $\overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n} A_{n,k}}$ by finite union $\bigcup_{k\in K_n^{\mathrm{fin}}} B_{n,k}$ where $\{B_{n,k} : k \in K_n^{\mathrm{fin}}\}$ is an ε-net (with $\varepsilon = 1/n$) for the compact set $\overline{\{A_{n,k} : k \in \mathbb{Z}\}}$. Similarly for τ_1 method.
- Define $U_n := \left(\Gamma^{n,\mathrm{fin}}_{\varepsilon''_n+\delta_n+3/n}(A_n) \;\; \cap \frac{1}{n}(\mathbb{Z} + \mathrm{i}\mathbb{Z})\right) \; + \; \frac{2}{n}\overline{\mathbb{D}}$.

$$\text{If} \qquad U_n = \Gamma^n_{\varepsilon_n''}(A) := \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon_n''}(A)}$$

then $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$, but only for **tridiagonal** A, and U_n can't be computed in finitely many operations.

Missing Ingredients (cf. Ben-Artzi et al. 2020)

- Realize that the entries of the tridiagonal matrix can themselves be square matrices - extends to A **banded**.
- Perturbation argument extends to A **band-dominated**, approximated by A_n (banded), with $\delta_n := \|A - A_n\| \to 0$.
- For τ method approximate $\overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon_n} A_{n,k}}$ by finite union $\bigcup_{k\in K_n^{\mathrm{fin}}} B_{n,k}$ where $\{B_{n,k} : k \in K_n^{\mathrm{fin}}\}$ is an ε-net (with $\varepsilon = 1/n$) for the compact set $\overline{\{A_{n,k} : k \in \mathbb{Z}\}}$. Similarly for τ_1 method.
- Define $U_n := \left(\Gamma^{n,\mathrm{fin}}_{\varepsilon_n''+\delta_n+3/n}(A_n) \ \cap \frac{1}{n}(\mathbb{Z}+\mathrm{i}\mathbb{Z})\right) + \frac{2}{n}\overline{\mathbb{D}}$. **Then** $\mathrm{Spec}\, A \subset U_n$**,** $U_n \to \mathrm{Spec}\, A$**, and** U_n **can be computed with finitely many operations.**

A final example [Feinberg/Zee 1999]

$$A = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence (Davies 2001)

A final example [Feinberg/Zee 1999]

$$A = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence (Davies 2001); i.e., **every finite pattern of ± 1's can be found somewhere in the infinite sequence b.**

A final example [Feinberg/Zee 1999]

$$A = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 1 & & & \\ & b_{-1} & 0 & 1 & & \\ & & b_0 & 0 & 1 & \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix},$$

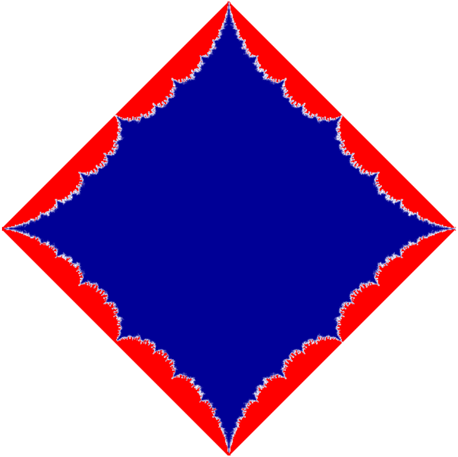
where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence (Davies 2001); i.e., **every finite pattern of ± 1's can be found somewhere in the infinite sequence b.**

This is an example where the τ method is convergent:

$$\operatorname{Spec} A \subset U_n := \bigcup_{k\in\mathbb{Z}} \operatorname{Spec}_{\varepsilon_n} A_{n,k} \to \operatorname{Spec} A, \quad \text{as} \quad n \to \infty,$$

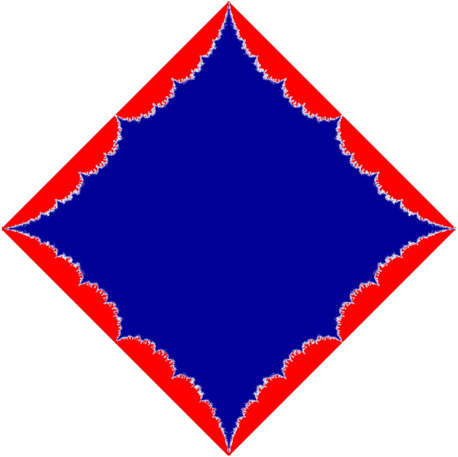
and where the union is finite: 2^{n-1} different matrices $A_{n,k}$.

Upper and lower bounds on $\mathrm{Spec}\, A$: which is sharp?



(The square has corners at ± 2 and $\pm 2\mathrm{i}$.)

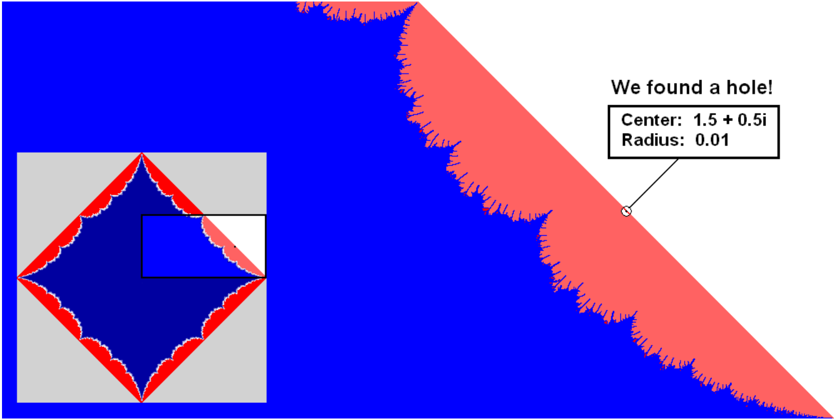
Upper and lower bounds on $\mathrm{Spec}\, A$: which is sharp?



(The square has corners at ± 2 and $\pm 2\mathrm{i}$.)

We have $\mathrm{Spec}\, A \subset U_n$ and $U_n \to \mathrm{Spec}\, A$ so, if $\lambda \notin \mathrm{Spec}\, A$, then $\lambda \notin U_n$ **for all sufficiently large** n.

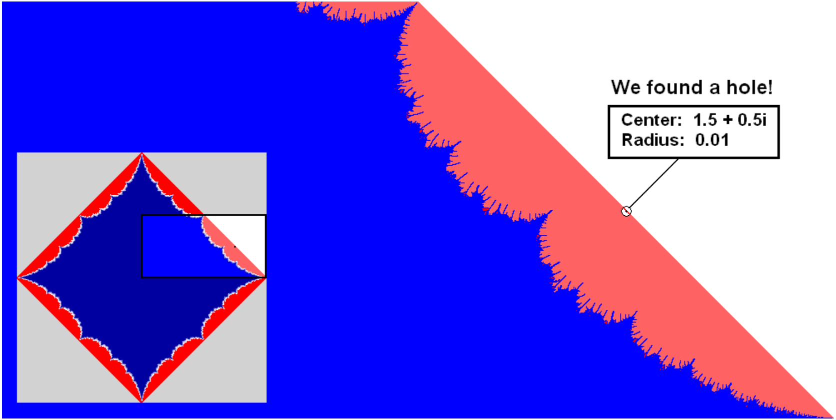
Is $\lambda = 1.5 + 0.5\text{i} \in \operatorname{Spec} A$?



$$\lambda = 1.5 + 0.5\text{i} \notin U_{34} \supset \operatorname{Spec} A, \quad \text{so} \quad \lambda \notin \operatorname{Spec} A,$$

so $\operatorname{Spec} A$ is a strict subset of the square.

Is $\lambda = 1.5 + 0.5\mathrm{i} \in \mathrm{Spec}\, A$?



$$\lambda = 1.5 + 0.5\mathrm{i} \;\notin\; U_{34} \;\supset\; \mathrm{Spec}\, A, \quad \text{so} \quad \lambda \notin \mathrm{Spec}\, A,$$

so $\mathrm{Spec}\, A$ is a strict subset of the square. This was a large calculation: we needed to check whether $2^{33} \approx 8.6 \times 10^9$ matrices of size 34×34 were positive definite!

Summary and conclusion

1. For **tridiagonal** A we have derived the τ, π, and τ_1 **inclusion set families** for $\mathrm{Spec}_\varepsilon A$, for $\varepsilon \geq 0$, i.e., for $n \in \mathbb{N}$,

$$\tau \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}$$

$$\pi \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon'_n} A^{\mathrm{per}}_{n,k}}$$

$$\tau_1 \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A) = \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)},$$

with explicit and optimsed formulae for $\varepsilon_n, \varepsilon'_n, \varepsilon''_n$. N.B. $\gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)$ can be interpreted as a pseudospectrum for a rectangular matrix.

Summary and conclusion

1. For **tridiagonal** A we have derived the τ, π, and τ_1 **inclusion set families** for $\mathrm{Spec}_\varepsilon A$, for $\varepsilon \geq 0$, i.e., for $n \in \mathbb{N}$,

$$\tau \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}$$

$$\pi \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon'_n} A^{\mathrm{per}}_{n,k}}$$

$$\tau_1 \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A) = \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)},$$

with explicit and optimsed formulae for $\varepsilon_n, \varepsilon'_n, \varepsilon''_n$. N.B. $\gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)$ can be interpreted as a pseudospectrum for a rectangular matrix.

2. Shown $\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, for $\varepsilon \geq 0$. N.B. $\mathrm{Spec}_0 A = \mathrm{Spec}\, A$.

1. For **tridiagonal** A we have derived the τ, π, and τ_1 **inclusion set families** for $\mathrm{Spec}_\varepsilon A$, for $\varepsilon \geq 0$, i.e., for $n \in \mathbb{N}$,

$$\tau \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}$$

$$\pi \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon'_n} A^{\mathrm{per}}_{n,k}}$$

$$\tau_1 \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A) = \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)},$$

with explicit and optimsed formulae for $\varepsilon_n, \varepsilon'_n, \varepsilon''_n$. N.B. $\gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)$ can be interpreted as a pseudospectrum for a rectangular matrix.

2. Shown $\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, for $\varepsilon \geq 0$. N.B. $\mathrm{Spec}_0 A = \mathrm{Spec}\, A$.

3. Shown some examples where the unions $\bigcup_{k\in\mathbb{Z}}$ are finite, exhibiting the inclusions and the τ_1-method convergence.

Summary and conclusion

1. For **tridiagonal** A we have derived the τ, π, and τ_1 **inclusion set families** for $\mathrm{Spec}_\varepsilon A$, for $\varepsilon \geq 0$, i.e., for $n \in \mathbb{N}$,

$$\tau \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon_n} A_{n,k}}$$

$$\pi \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \overline{\bigcup_{k\in\mathbb{Z}} \mathrm{Spec}_{\varepsilon+\varepsilon'_n} A^{\mathrm{per}}_{n,k}}$$

$$\tau_1 \text{ method:} \qquad \mathrm{Spec}_\varepsilon A \subset \Gamma^n_{\varepsilon+\varepsilon''_n}(A) = \overline{\bigcup_{k\in\mathbb{Z}} \gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)},$$

with explicit and optimsed formulae for $\varepsilon_n, \varepsilon'_n, \varepsilon''_n$. N.B. $\gamma^{n,k}_{\varepsilon+\varepsilon''_n}(A)$ can be interpreted as a pseudospectrum for a rectangular matrix.

2. Shown $\Gamma^n_{\varepsilon+\varepsilon''_n}(A) \to \mathrm{Spec}_\varepsilon A$ as $n \to \infty$, for $\varepsilon \geq 0$. N.B. $\mathrm{Spec}_0 A = \mathrm{Spec}\, A$.

3. Shown some examples where the unions $\bigcup_{k\in\mathbb{Z}}$ are finite, exhibiting the inclusions and the τ_1-method convergence.

4. Sketched extension to A band-dominated, and how τ_1-method can be adapted to need only finitely many operations while maintaining inclusion and convergence properties.

On Spectral Inclusion Sets and Computing the Spectra and Pseudospectra of Bounded Linear Operators

SIMON N. CHANDLER-WILDE, RATCHANIKORN CHONCHAIYA, and MARKO LINDNER

January 11, 2024

ABSTRACT. In this paper we derive novel families of inclusion sets for the spectrum and pseudospectrum of large classes of bounded linear operators, and establish convergence of particular sequences of these inclusion sets to the spectrum or pseudospectrum, as appropriate. Our results apply, in particular, to bounded linear operators on a separable Hilbert space that, with respect to some orthonormal basis, have a representation as a bi-infinite matrix that is banded or band-dominated. More generally, our results apply in cases where the matrix entries themselves are bounded linear operators on some Banach space. In the scalar matrix entry case we show that our methods, given the input information we assume, lead to a sequence of approximations to the spectrum, each element of which can be computed in finitely many arithmetic operations, so that, with our assumed inputs, the problem of determining the spectrum of a band-dominated operator has solvability complexity index one, in the sense of Ben-Artzi et al. (*C. R. Acad. Sci. Paris, Ser. I* **353** (2015), 931–936). As a concrete and substantial application, we apply our methods to the determination of the spectra of non-self-adjoint bi-infinite tridiagonal matrices that are pseudoergodic in the sense of Davies (*Commun. Math. Phys.* **216** (2001) 687–704).

Mathematics subject classification (2010): Primary 47A10; Secondary 47B36, 46E40, 47B80.
Keywords: band matrix, band-dominated matrix, solvability complexity index, pseudoergodic