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ON THE HALF-SPACE MATCHING METHOD FOR REAL WAVENUMBER*

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Abstract. The Half-Space Matching (HSM) method has recently been developed as a new method for the solution of two-dimensional scattering problems with complex backgrounds, providing an alternative to Perfectly Matched Layers (PML) or other artificial boundary conditions. Based on half-plane representations for the solution, the scattering problem is rewritten as a system coupling (1) a standard finite element discretization localized around the scatterer and (2) integral equations whose unknowns are traces of the solution on the boundaries of a finite number of overlapping half-planes contained in the domain. While satisfactory numerical results have been obtained for real wavenumbers, well-posedness and equivalence of this HSM formulation to the original scattering problem have been established for complex wavenumbers only. In the present paper we show, in the case of a homogeneous background, that the HSM formulation is equivalent to the original scattering problem also for real wavenumbers, and so is well-posed, provided the traces satisfy radiation conditions at infinity analogous to the standard Sommerfeld radiation condition. As a key component of our argument we show that if the trace on the boundary of a half-plane satisfies our new radiation condition, then the corresponding solution to the half-plane Dirichlet problem satisfies the Sommerfeld radiation condition in a slightly smaller half-plane. We expect that this last result will be of independent interest, in particular in studies of rough surface scattering.

Key words. Helmholtz equation, scattering, Sommerfeld radiation condition, integral equation, domain decomposition, rough surface scattering

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1. Introduction and the scattering problem.

1.1. The HSM method.

Recently, a new method called the Half-Space Matching (HSM) method has been developed as an (exact) artificial boundary condition for two-dimensional time-harmonic scattering problems. This method is based on explicit or semiexplicit expressions for the outgoing solutions of radiation problems in half-planes.

These expressions are established by using Fourier, generalized Fourier, or Floquet transforms when the background is, respectively, homogeneous [6, 5] (and possibly anisotropic [32, 3, 31]), stratified [27], or periodic [17, 18]. The domain exterior to a bounded region enclosing the scatterers is covered by a finite number $\mathcal{N}$ of half-planes (at least three). The unknowns of the formulation are the traces $\varphi^1, \ldots, \varphi^{\mathcal{N}}$ of the solution on the boundaries of these half-planes and the restriction of the solution to the bounded region. The system of equations which couples these unknowns is derived by writing compatibility conditions between the different representations of the solution. This coupled system includes second kind integral equations on the infinite boundaries of the half-planes.

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This new formulation is attractive and versatile as a method to truncate computational domains in problems of scattering by localized inhomogeneities in complex backgrounds, including backgrounds that may be different at infinity in different directions. It has been employed successfully in numerical implementations for various applications, like optical waveguides (including cases with different stratifications in different parts of the background domain) [27], or ultrasonic nondestructive testing (with an anisotropic elastic background) [32, 31].

Up to now the theoretical and numerical analysis of the method has remained an open question in the challenging, and practically relevant, nondissipative case when waves radiate out to infinity. But a rather complete analysis has been carried out in the simpler dissipative case, when the solution (and its traces) decay exponentially at infinity. In that case the analysis can be done using an L^2 framework for the traces: the associated formulation has been shown to be of Fredholm type and well-posed in a number of cases where the background is homogeneous (but not necessarily isotropic) [6, 5], with the sesquilinear form of the weak formulation coercive plus compact, enabling the numerical analysis of the method [5]. This analysis fails in the nondissipative case, not least because of the slow decay at infinity of the solution which results in non-L^2 traces.

1.2. Our main results. In this paper we address well-posedness of the HSM formulation in the nondissipative case for the scalar Helmholtz equation when the background is homogeneous and isotropic, so that

$$-\Delta u - k^2 u = 0 \tag{1}$$

outside some ball. (Here, assuming $\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence with $\omega > 0$, $k = \omega/c > 0$, and $c > 0$ are the constant wavenumber and wave speed, respectively, outside the ball.) For such configurations, it is well known that to achieve a well-posed scattering problem the scattered field u must satisfy the Sommerfeld radiation condition

$$\frac{\partial u(\boldsymbol{x})}{\partial r} - \mathrm{i}ku(\boldsymbol{x}) = o\left(r^{-1/2}\right) \quad \text{as } r := |\boldsymbol{x}| \to +\infty, \tag{2}$$

uniformly with respect to $\widehat{\boldsymbol{x}} := \boldsymbol{x}/r$. Further, it is well known that if u satisfies (1) outside some ball and (2), then

$$u(\boldsymbol{x}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{1/2}}\left(F(\widehat{\boldsymbol{x}}) + O(r^{-1})\right) \quad \text{as} \quad r \to \infty, \tag{3}$$

uniformly in $\widehat{\boldsymbol{x}} := \boldsymbol{x}/r$, where $F \in C^\infty(S^1)$, with S^1 the unit circle, is the so-called *far-field pattern* (e.g., [11, Lemma 2.5]).

The main results of this paper (Theorems 3.5 and 4.1) are to establish well-posedness for the HSM formulations, stated in detail as (30) and (57) below, in the case that $k > 0$ and the background is homogeneous and isotropic, so that the scattered field u satisfies (2) and (1) outside some ball. Precisely, we show that the HSM formulations are well-posed and equivalent to the original scattering problems provided we require that each half-plane trace φ^j is locally in L^2 and has the asymptotic behavior predicted by (3), meaning that

$$\varphi^j(\boldsymbol{x}) = c^j_\pm \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{1/2}}\left(1 + O(r^{-1})\right) \tag{4}$$

for some constants $c^j_\pm$, as $\boldsymbol{x}$ tends to infinity in the $\pm$ directions along the infinite boundary of the half-plane. Note that condition (4) can be seen as a radiation condition for the half-plane trace φ^j, playing the role in the HSM formulation for $k > 0$

that the Sommerfeld radiation condition (2) plays in the formulation of the original scattering problem.

Since (i) the HSM system of equations is derived from the unique solution to the scattering problem and (ii) the half-plane traces of the solution to the scattering problem do satisfy (4) (because (3) holds), we will see that existence of a solution holds by construction. The challenge remaining in order to show well-posedness is to establish uniqueness. There are two difficult aspects to this challenge. The first is to show that if φ^j satisfies (4) and u is the solution reconstructed from φ^j in the corresponding half-plane, so that u satisfies (1) in the half-plane and $u = \varphi^j$ on the half-plane boundary, then $u(\boldsymbol{x}) = O(r^{-1/2})$ as $r \to \infty$ and u satisfies an appropriate version of the Sommerfeld radiation condition (2). These properties are established in section 3.2. The second challenge, given that we assume ab initio only that the traces φ^j are locally L^2, is to show that each trace φ^j is locally in the trace space $H^{1/2}$, so that the reconstructed solution u is locally in H^1 as required. But this difficulty already arises in the corresponding formulation in the dissipative case, and the proof in that case is sketched in [6, section 3.3]. As preparation for the more difficult nondissipative case, we expand on that argument in section 2.2 below.

1.3. The significance of our results. The main significance of Theorems 3.5 and 4.1 is that these are the first well-posedness results in the important nondissipative case for the HSM method, a method which, as discussed above, has already proved effective for computing scattering by localized inhomogeneities in a range of complex backgrounds. Our theorems, challenging as they are to prove, are for the simplest case when the background is homogeneous and isotropic. However, we expect that these results and formulations will be an important stepping stone to more complex cases, and we discuss this further in concluding remarks to this paper.

Our main uniqueness result, Theorem 3.5, is also important because it is a crucial ingredient in the proof of well-posedness in [4, section 5] of the so-called *complex-scaled HSM method*, proposed recently in [4]. This new formulation is a version of the HSM method in the nondissipative case that, in the spirit of complex-scaling in Perfectly Matched Layers (PML) (e.g., [14]), achieves the L^2 framework of the dissipative case by analytically continuing the half-plane traces into the complex plane, so that the original half-plane boundary is replaced by a path in the complex plane on which the (analytically continued) trace is exponentially decreasing.

Additionally, we expect that the results that we establish (in Lemma 2.3 and section 3.2) on properties of the half-plane solution operator, which takes the trace on the boundary of a half-plane and recovers the half-plane solution, will be of independent interest. Indeed, there is large interest in so-called *rough surface scattering problems*, where the Helmholtz equation (or more complicated vector equations) are solved in a nonlocally perturbed half-plane D with boundary or transmission conditions on the rough surface ∂D (e.g., [13, 12, 1, 2, 24, 10, 28, 16, 33, 20]). In this context it is usual, to ensure uniqueness, to impose the so-called *upwards propagating radiation condition* (e.g., [12, 2, 24, 10, 16, 33]), which is precisely a requirement that, in some half-plane above the rough surface, the solution can be represented as the action of this half-plane solution operator on some L^∞ data on the boundary of the half-plane. This rough surface scattering application, in particular the use of the upwards propagating radiation condition, has driven significant study of this half-plane solution operator (e.g., [8, 9], [13, section 2], [2], [16, section 2.3]). The results in section 3.2 contribute to this study, shedding light on the relationship between the upwards propagating radiation condition and the Sommerfeld radiation condition, complementing previous

work, especially [13, Theorem 2.9] and [2, 20].

1.4. The scattering problem and the structure of this paper. Let us spell out in more detail the two-dimensional scattering problem that we consider in this paper. The propagation domain Ω is $\mathbb{R}^2$, or $\mathbb{R}^2$ minus a set of bounded Lipschitz obstacles so that Ω is a Lipschitz domain. We seek u that satisfies an inhomogeneous Helmholtz equation,

$$-\Delta u - k^2 \rho u = f \text{ in } \Omega. \tag{5}$$

Here ρ is a function in $L^\infty(\Omega)$ such that $\rho - 1$ is compactly supported, so that if ρ is real-valued and bounded below by a positive constant, (5) models propagation in a domain with a local perturbation in wave speed.[1] The source term $f \in L^2(\Omega)$ is also assumed to be compactly supported, so that (1) holds outside some ball. We suppose that the wavenumber is such that $k > 0$ in the nondissipative case, and such that $\Re(k) > 0$ and $\Im(k) > 0$ in the dissipative case. In the case when $\Omega \subsetneq \mathbb{R}^2$ we require also a standard (e.g., Dirichlet or Neumann) boundary condition on $\Gamma := \partial\Omega$. As is standard we seek a solution $u \in H^1(\Omega)$ in the dissipative case and $u \in H^1_{\text{loc}}(\Omega) := \{v_{|\Omega}; v \in H^1_{\text{loc}}(\mathbb{R}^2)\}$ in the case $k > 0$. In the nondissipative case we also require that u satisfies the Sommerfeld radiation condition (2).

To explain the HSM method, our uniqueness argument, and the role of the radiation condition (4), we consider two specific instances of the problem (5). To get the main ideas across and to prove the key uniqueness and well-posedness result, Theorem 3.5, we focus first on the case when $\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}$, for some convex polygon $\mathcal{O}$, with $\rho \equiv 1$ and $f \equiv 0$, so that (1) holds in Ω, imposing a Dirichlet boundary condition $u = g$ on Γ. In this case, where Ω is the exterior of a convex polygon and (1) holds in Ω, the HSM formulation is a system of second kind integral equations, in which the unknowns are the traces of u on the finite number of half-planes that abut the sides of the polygon $\mathcal{O}$. In section 2 we recall the HSM formulation for this problem in the dissipative case, in particular how uniqueness is proved. In section 3 we prove our main Theorem 3.5, i.e., we prove well-posedness in the nondissipative case under the additional radiation condition (4).

In section 4 we consider the more involved case where f is nonzero and/or $\rho \not\equiv 1$, in which case the second kind integral equation formulation is coupled to a local variational formulation in a bounded region Ω_b containing $\partial\Omega$ and the supports of f and $\rho - 1$. In this case we assume, to be specific, that the boundary condition on $\partial\Omega$ is a homogeneous Neumann condition. (The changes needed, to the formulation and well-posedness argument, to address other boundary conditions, and/or inhomogeneities in the boundary condition, indeed to include other types of compactly supported inhomogeneities, are straightforward.) The main result in this case, the uniqueness and well-posedness result Theorem 4.1, is proved by use of Theorem 3.5 and by adapting the proof of [4, Proposition 6.1]. Section 5 provides a brief conclusion and gives an indication of more complex configurations to which the same methods and arguments are expected to apply.

2. The HSM method for complex wavenumber.

2.1. The HSM formulation. In this section, as preparation for studying the HSM method for a real wavenumber, we first recall what is known about the method

[1] It is straightforward to incorporate more elaborate local behavior (e.g., local anisotropies, or local inhomogeneities in density as well as wave speed), as long as the resulting problem remains well-posed.

in the dissipative case. For this purpose, as discussed in section 1.4, we consider the Dirichlet problem for complex wavenumber ($\Im(k) > 0$, $\Re(k) > 0$) in the exterior Ω of a convex polygon $\mathcal{O}$. Thus, for given data $g \in H^{1/2}(\Gamma)$, where $\Gamma = \partial\Omega = \partial\mathcal{O}$, we consider the Dirichlet problem

$$\begin{cases} -\Delta u - k^2\, u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases} \tag{6}$$

It is well known that problem (6) has a unique solution $u \in H^1(\Omega)$.

If $\mathcal{O}$ is a polygon with $\mathcal{N}$ edges denoted $\Gamma^1, \ldots, \Gamma^{\mathcal{N}}$, the domain Ω is the union of $\mathcal{N}$ overlapping half-planes Ω^j, $j = 1, \ldots, \mathcal{N}$, that abut the $\mathcal{N}$ edges of the polygon $\mathcal{O}$ (see Figure 1). The angle between Γ^j and Γ^{j+1} is denoted as $\Theta^{j,j+1}$ or equivalently $\Theta^{j+1,j}$, where, here and in what follows, $j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}$, where $\mathbb{Z}/\mathcal{N}\mathbb{Z}$ is the ring of integers modulo $\mathcal{N}$. This convenient notation means that $j = 0$ is equivalent to $j = \mathcal{N}$ and $j = -1$ to $j = \mathcal{N} - 1$. Note finally that, because of the convexity, one has $0 < \Theta^{j,j+1} < \pi$ for all $j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}$.

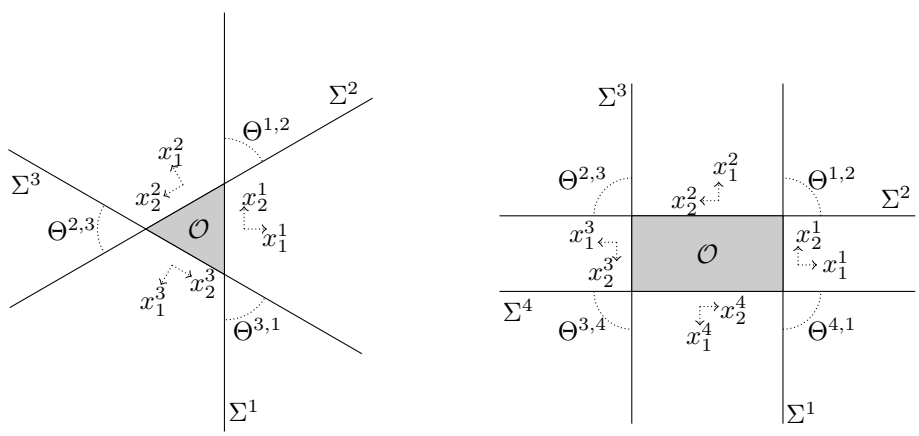

FIG. 1. *Examples of polygons $\mathcal{O}$ for $\mathcal{N} = 3$ and 4 and associated notation.*

It is convenient to make use of local coordinate systems $\boldsymbol{x}^j = (x_1^j, x_2^j)$ in each half-space Ω^j. The origin of all of them is the centroid O of the polygon $\mathcal{O}$. We define the Cartesian coordinate system (O, x_1^1, x_2^1) such that the axis Ox_1^1 is orthogonal to Γ^1 and directed into the exterior of the polygon, while the axis Ox_2^1 is $\pi/2$ counterclockwise to Ox_1^1. The other local coordinate systems (O, x_1^j, x_2^j) are defined recursively (see Figure 1) as follows:

$$\forall j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}, \quad \left| \begin{array}{l} x_1^{j+1} := \cos(\Theta^{j,j+1})\, x_1^j + \sin(\Theta^{j,j+1})\, x_2^j, \\ x_2^{j+1} := -\sin(\Theta^{j,j+1})\, x_1^j + \cos(\Theta^{j,j+1})\, x_2^j. \end{array} \right. \tag{7}$$

Defining a^j, for $j = 1, \ldots, \mathcal{N}$, to be the distance of the centroid of the polygon to the edge Γ^j, each half-plane Ω^j is defined in the local coordinate system (O, x_1^j, x_2^j) as

$$\Omega^j := \{x_1^j > a^j\} \times \{x_2^j \in \mathbb{R}\},$$

and its boundary, denoted by Σ^j, is given by

$$\Sigma^j := \{x_1^j = a^j\} \times \{x_2^j \in \mathbb{R}\}.$$

As explained in the introduction, the formulation uses the representation of the solution in each half-plane Ω^j in terms of its trace on Σ^j. More precisely, let us denote

$$\varphi^j := u\big|_{\Sigma^j} \quad \text{for } j = 1, \dots, \mathcal{N}, \tag{8}$$

so that

$$u\big|_{\Omega^j} = U^j(\varphi^j) \quad \text{for } j = 1, \dots, \mathcal{N}, \tag{9}$$

where, for $j = 1, \dots, \mathcal{N}$ and $\psi \in H^{1/2}(\Sigma^j)$, $U^j(\psi) \in H^1(\Omega^j)$ is the unique solution of

$$\left| \begin{array}{lll} -\Delta U^j - k^2 U^j = 0 & \text{in} & \Omega^j, \\ U^j = \psi & \text{on} & \Sigma^j, \end{array} \right. \tag{10}$$

this solution being well defined since $\Im(k) \neq 0$. We can express $U^j(\psi)$ explicitly in terms of its trace ψ using a Green's function representation:

$$U^j(\psi)(\boldsymbol{x}^j) = \int_{\Sigma^j} \frac{\partial G^j(\boldsymbol{x}^j, \boldsymbol{y}^j)}{\partial n(\boldsymbol{y}^j)} \, \psi(\boldsymbol{y}^j) \, \mathrm{d}s(\boldsymbol{y}^j), \quad \boldsymbol{x}^j \in \Omega^j, \tag{11}$$

where $G^j(\boldsymbol{x}^j, \boldsymbol{y}^j)$ is the Dirichlet Green's function for Ω^j and $n(\boldsymbol{y}^j)$ is the unit normal to Σ^j that points into Ω^j. Explicitly, $G^j(\boldsymbol{x}^j, \boldsymbol{y}^j) = \Phi(\boldsymbol{x}^j, \boldsymbol{y}^j) - \Phi(\widetilde{\boldsymbol{x}}^j, \boldsymbol{y}^j)$, with $\widetilde{\boldsymbol{x}}^j$ the image of $\boldsymbol{x}^j$ in Σ^j, where $\Phi(\boldsymbol{x}, \boldsymbol{y})$ is the standard fundamental solution of the Helmholtz equation defined by

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|\boldsymbol{x} - \boldsymbol{y}|), \quad \boldsymbol{x}, \, \boldsymbol{y} \in \mathbb{R}^2, \;\; \boldsymbol{x} \neq \boldsymbol{y}, \tag{12}$$

where $H_n^{(1)}$ is the Hankel function of the first kind of order n. This leads, finally, to

$$U^j(\psi)(\boldsymbol{x}^j) = 2 \int_{\Sigma^j} \frac{\partial \Phi(\boldsymbol{x}^j, \boldsymbol{y}^j)}{\partial y_1^j} \, \psi(\boldsymbol{y}^j) \, \mathrm{d}s(\boldsymbol{y}^j), \quad \boldsymbol{x}^j \in \Omega^j, \tag{13}$$

which can be rewritten as

$$U^j(\psi)(\boldsymbol{x}^j) = \int_{\mathbb{R}} \mathcal{H}(k; x_1^j - a^j, x_2^j - y_2^j) \, \psi(a^j, y_2^j) \, \mathrm{d}y_2^j, \quad \boldsymbol{x}^j = (x_1^j, x_2^j) \in \Omega^j, \tag{14}$$

where we have set

$$\mathcal{H}(k; x_1, x_2) := \frac{\mathrm{i}kx_1}{2} \frac{H_1^{(1)}(k|\boldsymbol{x}|)}{|\boldsymbol{x}|}. \tag{15}$$

To derive a system of equations whose unknowns are the traces φ^j of the solution, it suffices to observe that the half-plane representations must coincide where they coexist.

For instance, in the overlapping zone $\Omega^1 \cap \Omega^2$ (see Figure 2) we have

$$u = U^1(\varphi^1) = U^2(\varphi^2) \quad \text{in } \Omega^1 \cap \Omega^2, \tag{16}$$

and in particular,

$$\varphi^2 = U^1(\varphi^1) \quad \text{on } \Omega^1 \cap \Sigma^2, \tag{17}$$

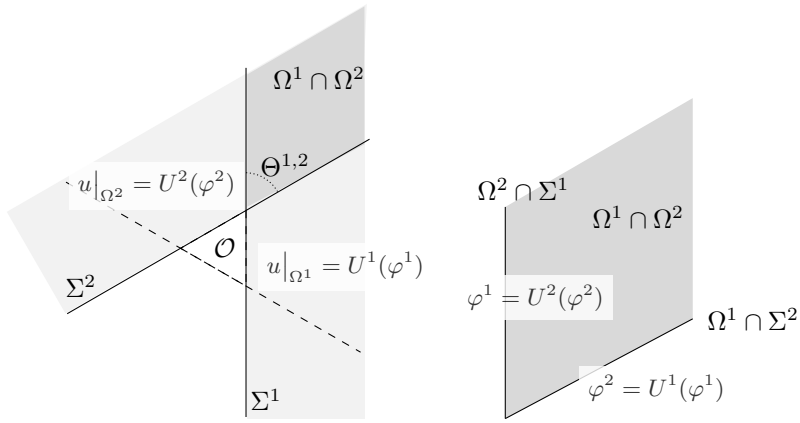

FIG. 2. *Compatibility condition in* $\Omega^1 \cap \Omega^2$.

which is a relation linking φ^1 and φ^2. By introducing the operator $D_{1,2} : L^2(\Sigma^1) \to L^2(\Omega^1 \cap \Sigma^2)$ defined by

$$D_{1,2}\,\psi := U^1(\psi)\big|_{\Omega^1\cap\Sigma^2}, \quad \psi \in L^2(\Sigma^1),$$

the relation (17) can be rewritten as

$$\varphi^2 = D_{1,2}\varphi^1 \quad \text{on } \Omega^1 \cap \Sigma^2. \tag{18}$$

From (16), we deduce similarly

$$\varphi^1 = U^2(\varphi^2) \quad \text{on } \Omega^2 \cap \Sigma^1,$$

another relation linking φ^1 and φ^2. Repeating this with 1 and 2 replaced, respectively, by j and $j+1$, we get $2\mathcal{N}$ equations linking the $\mathcal{N}$ traces. Let us introduce, for all $j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}$, the operators $D_{j,j\pm1} : H^{1/2}(\Sigma^j) \to H^{1/2}(\Omega^j \cap \Sigma^{j\pm1})$ defined by

$$D_{j,j\pm1}\,\psi := U^j(\psi)\big|_{\Omega^j\cap\Sigma^{j\pm1}}, \quad \psi \in H^{1/2}(\Sigma^j). \tag{19}$$

Then the compatibility relations between all the traces can be written as

$$\forall j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}, \quad \left| \begin{array}{ll} \varphi^j = D_{j-1,j}\,\varphi^{j-1} & \text{on } \Omega^{j-1} \cap \Sigma^j, \\ \varphi^j = D_{j+1,j}\,\varphi^{j+1} & \text{on } \Omega^{j+1} \cap \Sigma^j. \end{array} \right. \tag{20}$$

This system of equations has to be completed with the Dirichlet boundary condition, rewritten as

$$\varphi^j = g|_{\Gamma^j} \quad \text{on } \Gamma^j, \quad j = 1, \ldots, \mathcal{N}. \tag{21}$$

Note that for all j the operators $D_{j,j\pm1}$ can be given explicitly by using the representation (13) for U^j. They are integral operators and more precisely double-layer potential operators (in the sense, e.g., of [15, 7]).

One can easily check that (20)–(21) is equivalent to the original problem (6) and so is well-posed. We recall here the proof of this result because its general idea will be used in the results proved later in this paper.

THEOREM 2.1. *In the case where* $\Re(k) > 0$ *and* $\Im(k) > 0$*, there exists, for each* $g \in H^{1/2}(\Gamma)$*, a unique solution* $(\varphi^1, \dots, \varphi^{\mathcal{N}}) \in H^{1/2}(\Sigma^1) \times \cdots \times H^{1/2}(\Sigma^{\mathcal{N}})$ *of* (20)–(21).

Existence was proven by construction of (20)–(21) from the solution u of (6), with $\varphi^j := u|_{\Sigma^j}$ for $j = 1, \dots, \mathcal{N}$. Uniqueness follows from the following proposition since (6) is well-posed.

PROPOSITION 2.2. *Suppose* $\Re(k) > 0$ *and* $\Im(k) > 0$*, and let* $(\varphi^1, \dots, \varphi^{\mathcal{N}}) \in H^{1/2}(\Sigma^1) \times \cdots \times H^{1/2}(\Sigma^{\mathcal{N}})$ *be a solution to* (20)–(21) *(with* $g \in H^{1/2}(\Gamma)$*). Then, for all* $j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}$*,*

$$U^j(\varphi^j) = U^{j+1}(\varphi^{j+1}) \quad \text{in } \Omega^j \cap \Omega^{j+1}, \tag{22}$$

and the function defined by $u = U^j(\varphi^j)$ *in* Ω^j*,* $j = 1, \dots, \mathcal{N}$*, is the unique solution* $u \in H^1(\Omega)$ *of* (6).

Proof. To prove (22), let us set $v = U^j(\varphi^j) - U^{j+1}(\varphi^{j+1})$. By definition of the half-plane representations U^j and U^{j+1}, it is clear that $v \in H^1(\Omega^j \cap \Omega^{j+1})$ and that v satisfies $\Delta v + k^2 v = 0$ in $\Omega^j \cap \Omega^{j+1}$. Moreover, the compatibility conditions (20) imply that v vanishes on the boundary of $\Omega^j \cap \Omega^{j+1}$. Using that $\Im(k) \neq 0$, one deduces that $v = 0$ by uniqueness of the solution of the Dirichlet problem in $\Omega^j \cap \Omega^{j+1}$. The rest of the proof is straightforward. ∎

It has been shown in previous papers [6, 5] that, for both mathematical analysis and computation, it is more convenient to consider the HSM formulation in an L^2 framework, which means that φ^j is sought in $L^2(\Sigma^j)$ instead of $H^{1/2}(\Sigma^j)$ so that the formulation is

$$\text{Find } (\varphi^1, \dots, \varphi^{\mathcal{N}}) \in L^2(\Sigma^1) \times \cdots \times L^2(\Sigma^{\mathcal{N}}) \text{ such that for } j \in \mathbb{Z}/\mathcal{N}\mathbb{Z},$$
$$\left| \begin{array}{ll} \varphi^j = D_{j-1,j}\, \varphi^{j-1} & \text{on } \Sigma^j \cap \Omega^{j-1}, \\ \varphi^j = g|_{\Gamma^j} & \text{on } \Gamma^j, \\ \varphi^j = D_{j+1,j}\, \varphi^{j+1} & \text{on } \Sigma^j \cap \Omega^{j+1}. \end{array} \right. \tag{23}$$

Let us emphasize that this makes sense since $D_{j,j\pm1}\psi$ is well defined by (19) and (14) for all $\psi \in L^2(\Sigma^j)$ and the operators $D_{j,j\pm1}$ are, in fact, continuous from $L^2(\Sigma^j)$ to $L^2(\Omega^j \cap \Sigma^{j\pm1})$ [6]. It has been shown in [6] that problem (23) is of Fredholm type. More precisely, by rewriting it in an operator form, it is proven that the associated operator is Fredholm of index 0. This means that existence of a solution is equivalent to its uniqueness. We refer the reader to [6] for more detail. Here, we focus on the question of uniqueness. We need to prove that, for a solution of (23), $g = 0$ implies $\varphi^j = 0$ for all $j = 1, \dots, \mathcal{N}$. The sketch of the proof of this uniqueness result, which is much less straightforward than in the $H^{1/2}$ framework, has been given in [6]. We give a detailed presentation of this proof in the following paragraph, in a form that will be directly used for our main result, which is uniqueness for the case of a real wavenumber k.

2.2. The uniqueness result in an L^2 framework. The difficulty in proving a uniqueness result for problem (23) comes from the fact that the half-plane solutions $U^j(\varphi^j)$ (with $U^j(\varphi^j)$ defined now by (11), (13), or (14)) are, in general, not in $H^1(\Omega^j)$, assuming only that $\varphi^j \in L^2(\Sigma^j)$. But we will take advantage of the following lemma which summarizes key properties of the half-plane solutions with L^2 and/or L^∞ Dirichlet boundary data.

LEMMA 2.3. *Suppose that* $\Im(k) \geq 0, \Re(k) > 0$. *Let* $\phi \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$, *and define* $u(\boldsymbol{x})$, *for* $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2_+ := (0,\infty) \times \mathbb{R}$, *by*

$$u(\boldsymbol{x}) := \int_{\mathbb{R}} \mathcal{H}(k; x_1, x_2 - t)\phi(t)\,\mathrm{d}t, \tag{24}$$

where $\mathcal{H}$ *is defined by* (15). *Then the integral is well defined as a Lebesgue integral for* $\boldsymbol{x} \in \mathbb{R}^2_+$, $u \in C^\infty(\mathbb{R}^2_+)$, *and* $\Delta u + k^2 u = 0$ *in* $\mathbb{R}^2_+$. *Moreover, if* ϕ *is continuous on an open interval* $I \subset \mathbb{R}$, *then* $u \in C(\mathbb{R}^2_+ \cup \gamma)$, *where* $\gamma := \{(0,t);\ t \in I\}$, *and* $u(0,t) = \phi(t)$, $t \in I$. *Finally, if* $\phi \in L^2(\mathbb{R})$ *and* $\Im(k) > 0$, *then* $u \in L^2(\mathbb{R}^2_+)$ *and there exists a constant* $C_\infty > 0$ *independent of* ϕ *such that*

$$\|u\|_{L^2(\mathbb{R}^2_+)} \leq C_\infty \|\phi\|_{L^2(\mathbb{R})}, \tag{25}$$

whereas if $\Im(k) = 0$, *then for any* $L > 0$, $u \in L^2(\omega_L)$, *where* $\omega_L := \{\boldsymbol{x} \in \mathbb{R}^2_+;\ x_1 < L\}$, *and there exists a constant* $C_L > 0$ *independent of* ϕ *such that*

$$\|u\|_{L^2(\omega_L)} \leq C_L \|\phi\|_{L^2(\mathbb{R})}. \tag{26}$$

Proof. From asymptotics of the Hankel function $H_1^{(1)}$ for large and small arguments, it follows that, for every k with $\Im(k) \geq 0$, $\Re(k) > 0$, there exists a $C > 0$ such that

$$|\mathcal{H}(k; x_1, x_2)| \leq C \frac{x_1}{|\boldsymbol{x}|^2}(1 + |\boldsymbol{x}|^{1/2})e^{-\Im(k)|\boldsymbol{x}|}. \tag{27}$$

It follows that, for $\boldsymbol{x} \in \mathbb{R}^2_+$, the integral in (24) is well defined as a Lebesgue integral for any $\phi \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$. To see that $u \in C^\infty(\mathbb{R}^2_+)$ and satisfies the Helmholtz equation, it suffices to argue as in [9, Theorem 3.2]. If ϕ is continuous on an open interval $I \subset \mathbb{R}$, to see that $u \in C(\mathbb{R}^2_+ \cup \gamma)$ and that $u(0,t) = \phi(t)$ for $t \in I$, it is enough to show that $u \in C(\mathbb{R}^2_+ \cup \tilde{\gamma})$ and that $u(0,t) = \phi(t)$ for $t \in \tilde{I}$ for every compact $\tilde{\gamma} := \{(0,t);\ t \in \tilde{I}\} \subset \gamma$. So suppose $\tilde{I} \subset I$ is compact and write $\phi = \phi_1 + \phi_2$ where the support of ϕ_2 does not intersect $\tilde{I}$ and ϕ_1 is bounded and continuous. Correspondingly, we can split u as $u = u_1 + u_2$, where u_j is defined by (24) with ϕ replaced by ϕ_j. It follows from the definition that $u_2 \in C(\mathbb{R}^2_+ \cup \tilde{\gamma})$ with $u_2(0,t) = 0$ for $t \in \tilde{I}$, while $u_1 \in C(\overline{\mathbb{R}^2_+})$ and $u_1(0,t) = \phi_1(t)$ for $t \in \tilde{I}$ by [8, Theorem 3.1], so that $u(0,t) = \phi(t)$ for $t \in \tilde{I}$. Concerning the L^2 estimates (25) and (26) let us recall that for $\psi_1 \in L^1(\mathbb{R})$ and $\psi_2 \in L^2(\mathbb{R})$ the convolution product $\psi_1 * \psi_2$ belongs to $L^2(\mathbb{R})$, with the estimate (Young's convolution inequality)

$$\|\psi_1 * \psi_2\|_{L^2(\mathbb{R})} \leq \|\psi_1\|_{L^1(\mathbb{R})}\|\psi_2\|_{L^2(\mathbb{R})}.$$

For a given $x_1 > 0$, this implies that

$$\int_{\mathbb{R}} |u(x_1, x_2)|^2 dx_2 \leq \left[\int_{\mathbb{R}} |\mathcal{H}(k; x_1, x_2)| dx_2\right]^2 \|\phi\|^2_{L^2(\mathbb{R})}.$$

From (27), we deduce that, for some $C' > 0$,

$$\int_{\mathbb{R}} |\mathcal{H}(k; x_1, x_2)| dx_2 \leq C'(1 + x_1)^{1/2} e^{-\Im(k)x_1}, \quad x_1 > 0.$$

One easily obtains the estimates (25) for $\Im(k) > 0$ and (26) for $\Im(k) = 0$. ∎

Now to prove the uniqueness result, we need some results concerning L^2 solutions of the homogeneous Dirichlet problem in a domain with corners. We gather in the next lemma the results that we need which are proved in [19, Chapter 2] (see Theorem 2.3.3 and the proof of Theorem 2.3.7).

LEMMA 2.4. *Let Q be a bounded polygonal domain with N vertices denoted by $S_1, \dots, S_N$. Let us suppose that Q has $M < N$ re-entrant vertices that can be supposed to be $S_1, \dots, S_M$ without loss of generality. Let $P := \{S_1, \dots, S_N\}$ and suppose that $w \in C(\bar{Q} \setminus P) \cap C^2(Q)$ satisfies $\Delta w = 0$ in Q and $w = 0$ on $\partial Q \setminus P$. Then, if $w \in L^2(Q)$, there exist $\tilde{w} \in H^1(Q)$ and M complex constants $c_1, \dots, c_M$ such that*

$$w(\boldsymbol{x}) = \tilde{w}(\boldsymbol{x}) + \sum_{m=1}^{M} c_m \, r_m^{-\pi/\alpha_m} \sin(\pi\theta_m/\alpha_m), \quad \boldsymbol{x} \in Q,$$

where, for each m, α_m is the interior angle of Q at S_m and (r_m, θ_m) are the polar coordinates of $\boldsymbol{x}$ whose origin is S_m such that θ_m is the angle from one of the sides of ∂Q containing S_m. In particular, if $M = 0$ (Q is convex), then $w = 0$.

The previous lemma will be used to prove that particular L^2 solutions of the Helmholtz equations in unbounded domains with Dirichlet boundary conditions are in fact H^1 in every bounded subset of the domain. To prove that they are H^1 in the whole domain, we will resort to the following result, whose proof follows the arguments of [30, Lemma 2.2].

LEMMA 2.5. *Let $\Pi \subset \mathbb{R}^2$ be an unbounded open set such that $\Pi_R := \{\boldsymbol{x} \in \Pi; |\boldsymbol{x}| < R+1\}$ is a Lipschitz domain for all sufficiently large $R > 0$, and $v \in L^2(\Pi)$ a function such that $v_{|B} \in H^1(B)$ for all bounded open subsets B of Π. If v satisfies the Helmholtz equation $\Delta v + k^2 v = 0$ in Π and its trace is zero on $\partial\Pi$, then $v \in H^1(\Pi)$.*

Proof. Let us define the truncation function $\chi_R(\boldsymbol{x}) := F_R(|\boldsymbol{x}|)$, where $F_R \in C^\infty(\mathbb{R}^+)$ is chosen such that $0 \le F_R \le 1$, $F_R = 1$ on $[0, R]$, $F_R = 0$ on $[R+1, +\infty)$ and there exists a constant $c > 0$ independent of R such that $|F_R'| \le c\sqrt{F_R}$ (this can be achieved by defining F_R so that $F_R > 0$ on $(R, R+1)$ and vanishes quadratically at $R+1$). One has for all $R > 0$

$$\int_\Pi (\Delta v + k^2 v)\overline{v}\chi_R \, d\boldsymbol{x} = 0.$$

Since $v \in H^1(\Pi_R; \Delta) := \left\{u \in H^1(\Pi_R); \Delta u \in L^2(\Pi_R)\right\}$ we can apply Green's first identity in Π_R. Using that v vanishes on $\partial\Pi$, this yields

$$\int_\Pi |\nabla v|^2 \chi_R \, d\boldsymbol{x} \le k^2 \|v\|^2_{L^2(\Pi)} + \left| \int_\Pi \overline{v} \nabla v \cdot \nabla \chi_R \, d\boldsymbol{x} \right|.$$

Since $|\nabla \chi_R| \le c\sqrt{\chi_R}$, applying the Cauchy–Schwarz inequality gives

$$\int_\Pi |\nabla v|^2 \chi_R \, d\boldsymbol{x} \le k^2 \|v\|^2_{L^2(\Pi)} + c\|v\|_{L^2(\Pi)} \sqrt{\int_\Pi |\nabla v|^2 \chi_R \, d\boldsymbol{x}},$$

from which we deduce the existence of a constant $C > 0$ independent of R such that

$$\int_\Pi |\nabla v|^2 \chi_R \, d\boldsymbol{x} \le C\|v\|^2_{L^2(\Pi)}.$$

This being true for all R, we can conclude that $v \in H^1(\Pi)$. ∎

We are now able to prove the following well-posedness result.

THEOREM 2.6. *If $\Re(k) > 0$ and $\Im(k) > 0$, then there exists, for each $g \in H^{1/2}(\Gamma)$, a unique solution $(\varphi^1, \dots, \varphi^{\mathcal{N}}) \in L^2(\Sigma^1) \times \cdots \times L^2(\Sigma^{\mathcal{N}})$ of* (23).

This is a direct consequence of the following proposition and Theorem 2.1.

PROPOSITION 2.7. *Suppose $\Re(k) > 0$ and $\Im(k) > 0$, and let $(\varphi^1, \dots, \varphi^{\mathcal{N}}) \in L^2(\Sigma^1) \times \cdots \times L^2(\Sigma^{\mathcal{N}})$ be a solution of* (23) *(with $g \in H^{1/2}(\Gamma)$). Then $\varphi^j \in H^{1/2}(\Sigma^j)$, $j = 1, \dots, \mathcal{N}$.*

Proof. The proof consists of two steps.

1. Prove the relations (22) as in Proposition 2.2 but note that here the functions $(\varphi^1, \dots, \varphi^{\mathcal{N}})$ that solve (23) are a priori supposed to be only in L^2.
2. By noting that the traces on $\Sigma^1, \dots, \Sigma^{\mathcal{N}}$ of the unique solution of (6) form a solution of (23) (and are obviously in $H^{1/2}$), it suffices to prove that if $g = 0$, $\varphi^j = 0$ for $j = 1, \dots, \mathcal{N}$. To prove this uniqueness result, we show that, in the case that $(\varphi^1, \dots, \varphi^{\mathcal{N}})$ satisfies (23) with $g = 0$, the function defined (thanks to step 1) by $u = U^j(\varphi^j)$ in Ω^j ($U^j(\varphi^j)$ defined by (13)) is in $H^1(\Omega)$, with zero trace. Therefore, it is equal to 0 everywhere as the unique solution of (6) for $g = 0$. We finally deduce that each φ^j, which we show is the trace of u on Σ^j, is zero.

Step 1. As in the proof of Proposition 2.2, we introduce $v = U^j(\varphi^j) - U^{j+1}(\varphi^{j+1})$, where the $\varphi^j \in L^2(\Sigma^j)$ satisfy (23), in particular satisfy (20). A priori, we just know, thanks to Lemma 2.3, that $v \in L^2(\Omega^j \cap \Omega^{j+1}) \cap C^\infty(\Omega^j \cap \Omega^{j+1})$ and that v satisfies $\Delta v + k^2 v = 0$ in $\Omega^j \cap \Omega^{j+1}$. Note that it follows from (20) and Lemma 2.3 that the φ^j's are, in fact, continuous on $\Sigma^j \setminus \overline{\Gamma^j}$, so that, from Lemma 2.3, $U^j(\varphi^j) \in C(\overline{\Omega^j} \setminus \overline{\Gamma^j})$ and $U^j(\varphi^j) = \varphi^j$ on $\Sigma^j \setminus \overline{\Gamma^j}$. As a consequence, v is continuous in $\overline{\Omega^j \cap \Omega^{j+1}}$, except maybe at S_j, the intersection of Σ^j and Σ^{j+1}, and, thanks to (20), v vanishes on $\partial(\Omega^j \cap \Omega^{j+1}) \setminus \{S_j\}$. It follows by standard reflection and elliptic regularity arguments that $v \in C^\infty\big(\overline{\Omega^j \cap \Omega^{j+1}} \setminus \{S_j\}\big)$.

To conclude that $v = 0$, we need only prove that $v \in H^1(\Omega^j \cap \Omega^{j+1})$. First, we show that v is H^1 in all bounded subdomains of $\Omega^j \cap \Omega^{j+1}$. Let us introduce a bounded convex polygon $\mathcal{O}'$ such that $\overline{\mathcal{O}} \subset \mathcal{O}'$ (see Figure 3). Then $Q := \mathcal{O}' \cap \Omega^j \cap \Omega^{j+1}$ is a convex polygon. Let us consider the following Dirichlet problem: find $\tilde{v} \in H^1(Q)$ such that

$$\left| \begin{array}{ll} -\Delta \tilde{v} = k^2 v & \text{in } Q, \\ \tilde{v} = v & \text{on } \partial Q. \end{array} \right.$$

This problem has a unique solution as $v \in L^2(Q)$ and the restriction of v to ∂Q is in $H^{1/2}(\partial Q) \cap C(\partial Q)$, since $v \in C^\infty(\overline{Q} \setminus \{S_j\})$ and if we set $v = 0$ at S_j, $v|_{\partial Q}$ vanishes in a neighborhood of S_j. Further, by standard elliptic regularity arguments, $\tilde{v} \in C^2(Q)$ and (see [25, Corollary 7.11.7]) $\tilde{v} \in C(\bar{Q})$. One can finally apply Lemma 2.4 to the function $w = v - \tilde{v} \in C(\bar{Q} \setminus \{S_j\}) \cap C^2(Q) \cap L^2(Q)$. Since Q is convex, we conclude that $w = 0$, and consequently that $v|_Q = \tilde{v} \in H^1(Q)$. As $v \in C^\infty\big(\overline{\Omega^j \cap \Omega^{j+1}} \setminus \{S_j\}\big)$, we deduce that v is H^1 in all open bounded subdomains of $\Omega^j \cap \Omega^{j+1}$. Since also $v \in L^2(\Omega^j \cap \Omega^{j+1})$ satisfies the Helmholtz equation in $\Omega^j \cap \Omega^{j+1}$ and vanishes on $\partial(\Omega^j \cap \Omega^{j+1})$, Lemma 2.5 implies that $v \in H^1(\Omega^j \cap \Omega^{j+1})$. Thus $v = 0$ so that (22) holds.

Step 2. The second step consists of proving that if $g = 0$, the function defined by $u = U^j(\varphi^j)$ in Ω^j is equal to 0 everywhere, and deducing from this that each $\varphi^j = 0$. For $j = 1, \dots, \mathcal{N}$, as above let $S_j := \Sigma^j \cap \Sigma^{j+1}$ so that $P := \{S_1, S_2, \dots, S_{\mathcal{N}}\}$ is the set of corners of $\partial\Omega$ (i.e., the set of vertices of the polygon $\mathcal{O}$). Since, for

each j, φ^j is such that $\varphi^j = g = 0$ on Γ^j, and $\varphi^j \in C(\Sigma^j \setminus P)$ (see Step 1), it follows from Lemma 2.3 that $u \in C^\infty(\Omega^j) \cap L^2(\Omega^j) \cap C(\overline{\Omega^j} \setminus P)$ and $u = \varphi^j$ on $\Sigma^j \setminus P$. In particular, $u = \varphi^j = 0$ on Γ^j, and $\Delta u + k^2 u = 0$ in Ω^j. Consequently $u \in C^\infty(\Omega) \cap L^2(\Omega) \cap C(\overline{\Omega} \setminus P)$, $\Delta u + k^2 u = 0$ in Ω, and $u = 0$ on $\partial\Omega \setminus P$.

Again, to conclude that $u = 0$, we need to prove that $u \in H^1(\Omega)$. We proceed as in the first step of the proof. Introducing the bounded polygon $\Omega' := \mathcal{O}' \setminus \mathcal{O}$, we denote by $\tilde{u}$ the unique solution in $H^1(\Omega') \cap C(\overline{\Omega'}) \cap C^2(\Omega')$ of the Dirichlet problem

$$\left| \begin{array}{ll} -\Delta \tilde{u} = k^2 u & \text{in } \Omega', \\ \tilde{u} = u & \text{on } \partial\Omega'. \end{array} \right.$$

Then we apply Lemma 2.4 to the function $w = u - \tilde{u} \in C(\overline{\Omega'} \setminus P) \cap C^2(\Omega') \cap L^2(\Omega')$. But, in contrast to the argument in the first step, the polygon Ω' is not convex and has $\mathcal{N}$ re-entrant corners at the vertices $S_1, S_2, \ldots, S_{\mathcal{N}}$. Consequently, we deduce from Lemma 2.4 that there exist a function $\tilde{w} \in H^1(\Omega')$ and $\mathcal{N}$ complex coefficients $c_1, \ldots, c_{\mathcal{N}}$ such that

$$w = \tilde{w} + \sum_{j=1}^{\mathcal{N}} c_j \, r_j^{-\pi/\alpha_j} \sin(\pi\theta_j/\alpha_j) \quad \text{in } \Omega',$$

where, as explained in the lemma, α_j is the interior angle of Ω' at S_j and (r_j, θ_j) are polar coordinates centered at S_j. The definition of θ_j is such that $\sin(\pi\theta_j/\alpha_j)$ vanishes on $\Sigma^j \cap \partial\mathcal{O}$ and $\Sigma^{j+1} \cap \partial\mathcal{O}$, but note that it does not vanish on $\Sigma^j \cap \Omega^{j+1}$. Since $\varphi^j \in L^2(\Sigma^j)$, $\tilde{u} \in C(\overline{\Omega'})$, and $w = u - \tilde{u} = \varphi^j - \tilde{u}$ on $\Sigma^j \cap \Omega'$, we must have $w \in L^2(\Sigma^j \cap \Omega')$, which is possible only if $c_j = 0$. Indeed, since $\alpha_j < 2\pi$, we have $\pi/\alpha_j > 1/2$ so that $r_j^{-\pi/\alpha_j}$ is not square integrable near $r_j = 0$. Summing up, we have $w = \tilde{w}$ and $u = \tilde{u} + \tilde{w} \in H^1(\Omega')$. Since $u \in C^\infty(\Omega)$, one concludes that u is H^1 in all open bounded subdomains of Ω, and, finally, thanks to Lemma 2.5, that $u \in H^1(\Omega)$. We have shown that $u \in H^1(\Omega)$ and u satisfies (6) with $g = 0$. Thus $u = 0$ and since $\varphi^j = u$ on $\Sigma^j \setminus P$, we have that $\varphi^j = 0$ for all j, which ends the proof. ∎

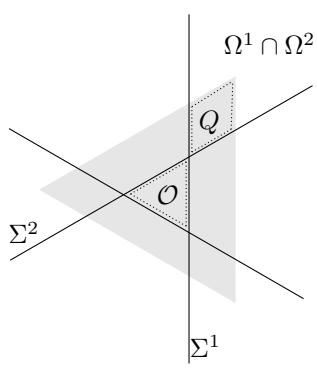

FIG. 3. *Illustration for the proof of Theorem* 2.6 *in the case* $j = 1$ *and* $\mathcal{N} = 3$. *The convex polygon* $\mathcal{O}'$ *is indicated in gray and the boundaries of* $\mathcal{O}$ *and* Q *with dotted contours.*

3. The HSM method for real wavenumber.

3.1. The HSM formulation. In this section we consider the Dirichlet problem of the previous section, but now with a real wavenumber $k > 0$. We seek $u \in H^1_{\text{loc}}(\Omega)$

such that

$$\begin{cases} -\Delta u - k^2\, u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma \end{cases} \tag{28}$$

for a given $g \in H^{1/2}(\Gamma)$, and such that u satisfies the Sommerfeld radiation condition (2), which it is convenient to rewrite as

$$\lim_{R\to+\infty} R^{1/2} \sup\left\{ \left| \frac{\partial u(\boldsymbol{x})}{\partial r} - \mathrm{i}ku(\boldsymbol{x}) \right| ; |\boldsymbol{x}| = R \right\} = 0. \tag{29}$$

It is well known that this problem has a unique solution.

Let us now give an HSM formulation of problem (28)–(29). If we set as before $\varphi^j := u\big|_{\Sigma^j}$ for $j = 1, \dots, \mathcal{N}$, one can easily check by a Green's identity argument that the half-plane representations (9), with $U^j(\varphi^j)$ given by (11) (equivalently, (13) or (14)), still hold. A main difference with the case of a complex wavenumber is that now the solution of (28)–(29) decays only slowly, like $r^{-1/2}$ as $r \to \infty$, so that we cannot expect that $\varphi^j \in L^2(\Sigma^j)$. However, $\varphi^j \in L^2(\Sigma^j) + L^\infty(\Sigma^j)$ so that, for $x_1^j > a^j$, the integral

$$\int_{\mathbb{R}} \mathcal{H}(k; x_1^j - a^j, x_2^j - y_2^j)\, \varphi^j(a^j, y_2^j)\, \mathrm{d}y_2^j$$

is still well defined by Lemma 2.3. Then one can check, exactly as in the case of a complex wavenumber, that the φ^j's satisfy the following HSM equations:

$$\left| \begin{array}{ll} \varphi^j = D_{j-1,j}\, \varphi^{j-1} & \text{on } \Sigma^j \cap \Omega^{j-1}, \\ \varphi^j = g|_{\Gamma^j} & \text{on } \Gamma^j, \\ \varphi^j = D_{j+1,j}\, \varphi^{j+1} & \text{on } \Sigma^j \cap \Omega^{j+1}, \end{array} \right. \qquad j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}, \tag{30}$$

where

$$D_{j,j\pm1}\, \varphi^j := U^j(\varphi^j)\big|_{\Omega^j \cap \Sigma^{j\pm1}}, \tag{31}$$

with $U^j(\varphi^j)$ given by (11). The idea is again, for numerical purposes, to replace problem (28)–(29) by the system of equations (30). But in contrast to the case with $\Im(k) > 0$, it is not clear in which space the functions φ^j's must be sought to make (30) uniquely solvable. In what follows, we will provide a framework which ensures existence and uniqueness of the solution of (30). More precisely, in this framework, the unique solution of (30) is given by $\varphi^j := u\big|_{\Sigma^j}$, where u is the unique solution of (28)–(29).

Let us first recall, as just remarked above, that $U^j(\varphi)$ is well defined when $\varphi \in L^2(\Sigma^j) + L^\infty(\Sigma^j)$ by Lemma 2.3, and, indeed, we will require that the solution of (30) satisfies $\varphi^j \in L^2(\Sigma^j) + L^\infty(\Sigma^j)$ for $j = 1, \dots, \mathcal{N}$. But, as discussed in section 1.2, our uniqueness result requires an additional property: analogously to requiring that $u \in H^1_{\mathrm{loc}}(\Omega)$ satisfies the Sommerfeld radiation condition when k is real, we will require, for $j = 1, \dots, \mathcal{N}$, that φ^j has the following asymptotics playing the role of a radiation condition:

$$\varphi^j(a^j, t) = \begin{cases} c_+^j\, \mathrm{e}^{\mathrm{i}k|t|} |t|^{-1/2} \left(1 + O(|t|^{-1})\right) & \text{as} \quad t \to +\infty, \\ c_-^j\, \mathrm{e}^{\mathrm{i}k|t|} |t|^{-1/2} \left(1 + O(|t|^{-1})\right) & \text{as} \quad t \to -\infty \end{cases} \tag{32}$$

for some constants $c_\pm^j \in \mathbb{C}$. This obviously holds when $\varphi^j := u\big|_{\Sigma^j}$, where u is the unique solution of (28)–(29), since, as discussed in section 1.2, u satisfies (3).

The main result of the present paper is Theorem 3.5 stating that the radiation condition (32) is strong enough to ensure uniqueness for (30) for k real. We will prove it by following the same steps as in the case of a complex wavenumber, proving first the compatibility of half-plane representations in the overlaps between half-planes, and then uniqueness for the reconstructed solution in the exterior domain Ω. To establish these results, we will use classical uniqueness results for the Helmholtz equation on unbounded domains when the solution satisfies the Sommerfeld radiation condition at infinity. A preliminary question is as follows: Can we prove, using (32), that the half-plane representations (14) satisfy a Sommerfeld condition at infinity? We will establish in the next subsection a slightly weaker result which will be sufficient for the proof of Theorem 3.5.

3.2. Properties of the half-plane solution. We consider in this subsection the half-space $\mathbb{R}^2_+ := (0,\infty)\times\mathbb{R}$. Let us prove a preliminary lemma for the solution of the Dirichlet problem in $\mathbb{R}^2_+$ given by (24) in the case that the Dirichlet data decays sufficiently rapidly at infinity.

LEMMA 3.1. *Suppose that $\phi \in L^\infty(\mathbb{R})$ satisfies, for some $p > 1$,*

$$|\phi(t)| \le (1+|t|)^{-p}, \quad t \in \mathbb{R}, \tag{33}$$

and define $u(\boldsymbol{x})$, for $\boldsymbol{x} \in \mathbb{R}^2_+$, by

$$u(\boldsymbol{x}) := \int_{\mathbb{R}} \mathcal{H}(k; x_1, x_2 - t)\phi(t)\,\mathrm{d}t. \tag{34}$$

Then there exists a constant $C(p) > 0$ such that

$$|u(\boldsymbol{x})| \le C(p)(1+|\boldsymbol{x}|)^{-1/2}, \quad \boldsymbol{x} \in \mathbb{R}^2_+. \tag{35}$$

Proof. We will use the bound (27), where $C > 0$, there and throughout the remainder of the proof, denotes a constant independent of $\boldsymbol{x}$ (and also independent of p), not necessarily the same at each occurrence. It will be convenient for the proof to denote by $u[\phi]$ the function u defined by (34). Then by defining $\chi_{x_2} \in L^\infty(\mathbb{R})$, for $x_2 \in \mathbb{R}$, to be the characteristic function of the interval

$$I_{x_2} := \{t \in \mathbb{R}; |x_2 - t| \le (1+|x_2|)/2\},$$

we have $u[\phi] = u[\chi_{x_2}\phi] + u[(1-\chi_{x_2})\phi]$. We will establish the bound (35) for each term of the sum (cf. the argument between (4.16) and (4.17) in [1]).

For the first term we have

$$|u[\chi_{x_2}\phi](\boldsymbol{x})| \le \int_{\mathbb{R}} |\mathcal{H}(k; x_1, x_2 - t)|\chi_{x_2}(t)|\phi(t)|\mathrm{d}t,$$

which gives

$$|u[\chi_{x_2}\phi](\boldsymbol{x})| \le 2\|\chi_{x_2}\phi\|_{L^\infty(\mathbb{R})} \int_0^{\frac{1+|x_2|}{2}} |\mathcal{H}(k; x_1, s)|\mathrm{d}s. \tag{36}$$

Now note that if $|t| < |x_2|/3$ and $|x_2| \ge 3$, then $|t - x_2| > 2|x_2|/3 \ge (1+|x_2|)/2$, so that $\chi_{x_2}(t) = 0$. Thus, for $|x_2| \ge 3$, (33) implies that

$$\|\chi_{x_2}\phi\|_{L^\infty(\mathbb{R})} \le \sup_{|t|\ge|x_2|/3} |\phi(t)| \le (1+|x_2|/3)^{-p},$$

while also $\|\chi_{x_2}\phi\|_{L^\infty(\mathbb{R})} \le \|\phi\|_{L^\infty(\mathbb{R})} \le 1$, for every $x_2 \in \mathbb{R}$, so that

$$\|\chi_{x_2}\phi\|_{L^\infty(\mathbb{R})} \le C(1+|x_2|)^{-p}, \quad x_2 \in \mathbb{R}. \tag{37}$$

Moreover, it follows from (27) that

$$\begin{aligned}\int_0^{\frac{1+|x_2|}{2}} |\mathcal{H}(k;x_1,s)|\mathrm{d}s &\le Cx_1 \int_0^{\frac{1+|x_2|}{2}} \left[(x_1^2+s^2)^{-1} + (x_1^2+s^2)^{-3/4}\right]\mathrm{d}s \\ &= C\int_0^{\frac{1+|x_2|}{2x_1}} \left[(1+t^2)^{-1} + x_1^{1/2}(1+t^2)^{-3/4}\right]\mathrm{d}t \\ &\le C(1+x_1)^{1/2}\int_0^{\frac{1+|x_2|}{2x_1}} (1+t^2)^{-3/4}\mathrm{d}t \\ &\le C(1+x_1)^{1/2}\min\left(1, \frac{1+|x_2|}{2x_1}\right).\end{aligned}$$

Combining this with (36) and (37), we get the following estimate:

$$|u[\chi_{x_2}\phi](\boldsymbol{x})| \le C(1+|x_2|)^{-p}(1+x_1)^{1/2}\min\left(1, \frac{1+|x_2|}{2x_1}\right),$$

which gives the expected bound (recalling that $p > 1$)

$$|u[\chi_{x_2}\phi](\boldsymbol{x})| \le C(1+|\boldsymbol{x}|)^{-1/2}, \quad \boldsymbol{x} \in \mathbb{R}^2_+.$$

To see that the same bound holds for $u[(1-\chi_{x_2})\phi]$, we note that

$$|u[(1-\chi_{x_2})\phi](\boldsymbol{x})| \le C\sup_{t\in\mathbb{R}} |\mathcal{H}(k;x_1,x_2-t)(1-\chi_{x_2})(t)| \int_{\mathbb{R}} |\phi(t)|\,\mathrm{d}t.$$

Observing that, for $\boldsymbol{x} \in \mathbb{R}^2_+$ and $t \in \mathbb{R}\setminus I_{x_2}$,

$$(x_1^2 + (x_2-t)^2)^{1/2} \ge (x_1^2 + (1+|x_2|)^2/4)^{1/2} \ge C(1+|\boldsymbol{x}|),$$

and using (27) and (33), it follows that, for $\boldsymbol{x} \in \mathbb{R}^2_+$,

$$|u[(1-\chi_{x_2})\phi](\boldsymbol{x})| \le C\frac{x_1}{(1+|\boldsymbol{x}|)^{3/2}}\int_{\mathbb{R}} |\phi(t)|\,\mathrm{d}t \le \frac{C}{p-1}(1+|\boldsymbol{x}|)^{-1/2}. \tag{38}$$ ∎

It follows from standard interior elliptic regularity results that if $u \in C^2(\mathbb{R}^2_+)$ satisfies

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2_+,$$

then for every $\varepsilon > 0$ and $\boldsymbol{x} \in \mathbb{R}^2_+$ with $x_1 > \varepsilon$, so that $\{\boldsymbol{y} \in \mathbb{R}^2; |\boldsymbol{x}-\boldsymbol{y}| \le \varepsilon\} \subset \mathbb{R}^2_+$,

$$|\nabla u(\boldsymbol{x})| \le C\,\frac{1+k^2\varepsilon^2}{\varepsilon}\max_{|\boldsymbol{x}-\boldsymbol{y}|\le\varepsilon} |u(\boldsymbol{y})|, \tag{39}$$

where C is an absolute constant (see, e.g., [13, Lemma 2.7]). Combining this estimate with the previous lemma, we immediately have the following corollary.

COROLLARY 3.2. *Suppose that $\phi \in L^\infty(\mathbb{R})$ satisfies* (33) *for some $p > 1$. Then there exists a constant $C(p) > 0$ such that for all $\varepsilon > 0$, the function u defined by* (34) *satisfies*

$$|\nabla u(\boldsymbol{x})| \le C(p)\,\frac{1+k^2\varepsilon^2}{\varepsilon}(1+|\boldsymbol{x}|)^{-1/2}, \quad x_1 > \varepsilon,\ x_2 \in \mathbb{R}. \tag{40}$$

Now, using the previous results, we will establish a radiation condition for u defined by (34) when its trace φ satisfies itself a radiation condition.

LEMMA 3.3. *Suppose that $\phi \in C(\mathbb{R})$ satisfies*

$$\phi(t) = \begin{cases} c_+ \, \mathrm{e}^{\mathrm{i}k|t|}|t|^{-1/2} + O(|t|^{-p}) & \text{as} \quad t \to +\infty, \\ c_- \, \mathrm{e}^{\mathrm{i}k|t|}|t|^{-1/2} + O(|t|^{-p}) & \text{as} \quad t \to -\infty \end{cases} \tag{41}$$

for some constants $c_+, c_- \in \mathbb{C}$, and $p > 1$. Define u by (34) in $\mathbb{R}^2_+$. Then the following properties hold:

(i) *there exists a constant $C > 0$ such that*

$$|u(\boldsymbol{x})| \le C(1 + |\boldsymbol{x}|)^{-1/2}, \quad \boldsymbol{x} \in \mathbb{R}^2_+; \tag{42}$$

(ii) *for all $\varepsilon > 0$,*

$$\lim_{R \to +\infty} R^{1/2} \sup \left\{ \left| \frac{\partial u(\boldsymbol{x})}{\partial r} - \mathrm{i}ku(\boldsymbol{x}) \right| ; x_1 > \varepsilon \text{ and } |\boldsymbol{x}| = R \right\} = 0. \tag{43}$$

Proof. We use the same notation $u[\phi]$ for (34) as in the proof of Lemma 3.1. We prove this proposition by writing ϕ as $\phi = \phi^{(1)} + \phi^{(2)}$, and correspondingly writing the integral (34) as $u[\phi] = u[\phi^{(1)}] + u[\phi^{(2)}]$, choosing this splitting as follows: $\phi^{(1)}$ is the trace on the boundary of a solution of the Helmholtz equation in $\mathbb{R}^2_+$ that clearly satisfies (42) and (43), and $\phi^{(2)}(t)$ decays sufficiently rapidly as $|t| \to \infty$ so that, thanks to Lemma 3.1 and Corollary 3.2, $u[\phi^{(2)}]$ also satisfies (42) and (43).

In more detail, from the asymptotics of the Hankel function $H_0^{(1)}$ it follows that, for some nonzero $c^* \in \mathbb{C}$,

$$\Phi(\boldsymbol{x}, (0, t)) = c^* \, \mathrm{e}^{\mathrm{i}k|t - x_2|} \, |t|^{-1/2} + O(|t|^{-3/2}) \quad \text{as} \quad |t| \to \infty \tag{44}$$

for every $\boldsymbol{x} \in \mathbb{R}^2$. Pick $\boldsymbol{z} = (z_1, z_2), \boldsymbol{z}' = (z_1', z_2') \in \mathbb{R}^2$ with $k(z_2 - z_2')/\pi \notin \mathbb{Z}$, and with $z_1, z_1' < 0$. Define $\phi^{(1)} \in C(\mathbb{R})$ by

$$\phi^{(1)}(t) := (c/c^*)\Phi(\boldsymbol{z}, (0, t)) + (c'/c^*)\Phi(\boldsymbol{z}', (0, t)) \text{ for } t \in \mathbb{R}. \tag{45}$$

Then $\phi^{(1)}(t)$ has the same leading asymptotic behavior as $\phi(t)$ as $t \to \pm\infty$ provided

$$c\mathrm{e}^{-\mathrm{i}kz_2} + c'\mathrm{e}^{-\mathrm{i}kz_2'} = c_+ \quad \text{and} \quad c\mathrm{e}^{\mathrm{i}kz_2} + c'\mathrm{e}^{\mathrm{i}kz_2'} = c_-. \tag{46}$$

Let us choose c and c' so that these two linear constraints hold; this is possible since $k(z_2 - z_2')/\pi \notin \mathbb{Z}$ implies that the determinant $\exp(\mathrm{i}k(z_2' - z_2)) - \exp(\mathrm{i}k(z_2 - z_2') = 2\mathrm{i}\sin(k(z_2 - z_2')) \neq 0$. Then it is clear from an application of Green's second identity that, for $\boldsymbol{x} \in \mathbb{R}^2_+$, $u[\phi^{(1)}](\boldsymbol{x}) = (c/c^*)\Phi(\boldsymbol{z}, \boldsymbol{x}) + (c'/c^*)\Phi(\boldsymbol{z}', \boldsymbol{x})$ (see, e.g., the proof that (ii)⇒(v)⇒(i) in [13, Theorem 2.9]). Hence $u[\phi^{(1)}]$ satisfies items (i) and (ii).

It remains to show that the same is true for $u[\phi^{(2)}]$ where $\phi^{(2)} = \phi - \phi^{(1)}$. It follows from (41) and (44)–(46) that, for some $c_0 > 0$, where $p' := \min(p, 3/2)$,

$$|\phi^{(2)}(t)| \le c_0(1 + |t|)^{-p'}, \quad t \in \mathbb{R}. \tag{47}$$

Item (i) is, therefore, a direct consequence of Lemma 3.1. To see that $u[\phi^{(2)}]$ also satisfies the radiation condition (ii), choose $q \in (1, p')$ and, for each $A > 0$, let χ_A be the characteristic function of the interval $[-A, A]$. Write $u[\phi^{(2)}] = u[\chi_A \phi^{(2)}] + u[(1 - \chi_A)\phi^{(2)}]$. Now, by (47),

$$|(1 - \chi_A(t))\phi^{(2)}(t)| \le c_0(1 + |t|)^{-q}(1 + A)^{q - p'}$$

so that, by (35) and (40) (applied with p replaced by q), there exists a constant $C(q) > 0$ depending only on q such that

$$|u[(1-\chi_A)\phi^{(2)}](\boldsymbol{x})| \le C(q)c_0(1+A)^{q-p'}(1+|\boldsymbol{x}|)^{-1/2}, \quad \boldsymbol{x} \in \mathbb{R}^2_+,$$

and from Corollary 3.2, for all $\varepsilon > 0$, there exists a constant $C(q,\varepsilon) > 0$ depending only on q and ε such that

$$|\nabla u[(1-\chi_A)\phi^{(2)}](\boldsymbol{x})| \le C(q,\varepsilon)c_0(1+A)^{q-p'}(1+|\boldsymbol{x}|)^{-1/2}, \quad x_1 > \varepsilon,\, x_2 \in \mathbb{R}.$$

Let us set for any function v defined on $\mathbb{R}^2_+$

$$M_{R,\varepsilon}(v) := R^{1/2} \sup\left\{ \left| \frac{\partial v(\boldsymbol{x})}{\partial r} - \mathrm{i}kv(\boldsymbol{x}) \right|; x_1 > \varepsilon \text{ and } |\boldsymbol{x}| = R \right\}.$$

It follows from the estimates above that, for all $R > 0$,

$$M_{R,\varepsilon}(u[(1-\chi_A)\phi^{(2)}]) \le \max(C(q), C(q,\varepsilon))c_0(1+A)^{q-p'}.$$

Thus, for all $\varepsilon > 0$, given $\eta > 0$ we can choose $A > 0$ such that $M_{R,\varepsilon}(u[(1-\chi_A)\phi^{(2)}]) \le \eta/2$ for all $R > 0$. But also, for each $A > 0$, it is standard that $u[\chi_A\phi^{(2)}](\boldsymbol{x})$, a double-layer potential supported on a bounded interval, satisfies the Sommerfeld radiation condition, in particular that $M_{R,\varepsilon}(u[\chi_A\phi^{(2)}]) \le \eta/2$ for R large enough. Thus, $M_{R,\varepsilon}(u[\phi^{(2)}]) \le \eta$ for all sufficiently large R, which concludes the proof. ∎

COROLLARY 3.4. *Suppose that $\phi = \phi_1 + \phi_2$, where $\phi_1 \in L^2(\mathbb{R})$ is compactly supported and $\phi_2 \in C(\mathbb{R})$. If ϕ satisfies* (41) *and u is defined by* (34)*, then*

(i) *there exist two constants $C > 0$ and $R > 0$ such that*

$$|u(\boldsymbol{x})| \le C(1+|\boldsymbol{x}|)^{-1/2}, \quad \boldsymbol{x} \in \mathbb{R}^2_+,\ |\boldsymbol{x}| > R; \tag{48}$$

(ii) *for all $\varepsilon > 0$,* (43) *holds.*

Proof. Again, we write $u[\phi] = u[\phi_1] + u[\phi_2]$. The properties for $u[\phi_2]$ have been established in the previous lemma, while $u[\phi_1]$ is a double-layer potential supported on a bounded interval for which such properties are standard, e.g., [11, Theorem 2.14, Lemma 2.5]. ∎

3.3. Proof of the uniqueness result. We are now able to prove the first main result of this paper.

THEOREM 3.5. *Suppose $k > 0$. There exists, for each $g \in H^{1/2}(\Gamma)$, a unique solution $(\varphi^1, \ldots, \varphi^{\mathcal{N}}) \in L^2_{\mathrm{loc}}(\Sigma^1) \times \cdots \times L^2_{\mathrm{loc}}(\Sigma^{\mathcal{N}})$ satisfying* (30) *such that, for all j, φ^j satisfies the radiation condition* (32)*.*

Existence was proven by the construction above of a solution of (30) from the unique solution u of (28), (2), namely $\varphi^j := u|_{\Sigma^j}$, $j = 1, \ldots, \mathcal{N}$. Note, in particular, that the Sommerfeld radiation condition (2) for u implies the far-field asymptotics (3), which implies in turn that each trace φ^j satisfies the radiation condition (32). Uniqueness follows from the following proposition since (28), (2) is well-posed.

PROPOSITION 3.6. *Suppose $k > 0$, and let $(\varphi^1, \ldots, \varphi^{\mathcal{N}}) \in L^2_{\mathrm{loc}}(\Sigma^1) \times \cdots \times L^2_{\mathrm{loc}}(\Sigma^{\mathcal{N}})$ satisfying* (30) *(with $g \in H^{1/2}(\Gamma)$) be such that, for all j, φ^j satisfies the radiation condition* (32)*. Then the compatibility equations* (22) *hold for all $j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}$, the function defined by $u = U^j(\varphi^j)$ in Ω^j, $j = 1, \ldots, \mathcal{N}$, is the unique solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of* (28), (2)*, and $\varphi^j = u|_{\Sigma^j}$ for each j.*

Proof. For each j, φ^j can be decomposed as $\varphi^j = \varphi_1^j + \varphi_2^j$ with $\varphi_1^j = \varphi^j \chi_A^j$ and $\varphi_2^j = \varphi^j(1-\chi_A^j)$, χ_A^j being the characteristic function of $\{(x_1^j = a^j, x_2^j); x_2^j \in (-A, A)\}$. Since $\varphi^j \in L^2_{\mathrm{loc}}(\Sigma^j)$ satisfies (32), we have that $\varphi_1^j \in L^2(\Sigma^j)$ with compact support and, for A large enough, $\varphi_2^j \in L^\infty(\Sigma^j)$. From Lemma 2.3, and by the definition (31) of the operator $D_{j,j\pm1}$, since $(\varphi^1, \dots, \varphi^{\mathcal{N}})$ satisfies (30), we deduce, provided we choose A large enough, that $\varphi_2^j \in C(\Sigma^j)$ for all j.

We follow now the same steps as in the proof of Proposition 2.7.

The first step consists of proving equations (22). Again, we introduce $v = U^j(\varphi^j) - U^{j+1}(\varphi^{j+1})$ and one can show by using Lemma 2.3 as in the proof of Proposition 2.7 that $v \in L^2_{\mathrm{loc}}(\Omega^j \cap \Omega^{j+1}) \cap C^\infty(\Omega^j \cap \Omega^{j+1}) \cap C(\overline{\Omega^j \cap \Omega^{j+1}} \setminus \{S_j\})$, satisfies $\Delta v + k^2 v = 0$ in $\Omega^j \cap \Omega^{j+1}$, and vanishes on $\partial(\Omega^j \cap \Omega^{j+1}) \setminus \{S_j\}$. Moreover, proceeding exactly as in the proof of Proposition 2.7, one can show that $v \in H^1_{\mathrm{loc}}(\Omega^j \cap \Omega^{j+1})$. We would like, by applying Corollary 3.4 to $U^j(\varphi^j)$ and $U^{j+1}(\varphi^{j+1})$, to deduce that v also satisfies the Sommerfeld radiation condition in $\Omega^j \cap \Omega^{j+1}$. The issue is that Corollary 3.4 does not ensure that v satisfies the Sommerfeld radiation condition up to the boundary. But this difficulty can be overcome; since $v = 0$ on $\partial(\Omega^j \cap \Omega^{j+1})$, reflection arguments and standard elliptic regularity results combined with the estimate (48) imply that, for R large enough, there exists a constant $C > 0$ such that

$$|v(\boldsymbol{x})| + |\nabla v(\boldsymbol{x})| \le C(1 + |\boldsymbol{x}|)^{-1/2}, \quad \boldsymbol{x} \in \Omega^j \cap \Omega^{j+1}, |\boldsymbol{x}| > R. \tag{49}$$

Then, denoting $\Sigma_R^j := \{\boldsymbol{x} \in \Omega^j \cap \Omega^{j+1}; |\boldsymbol{x}| = R\}$ and

$$d^j(\boldsymbol{x}) := \min_{\boldsymbol{y} \in \partial(\Omega^j \cap \Omega^{j+1})} |\boldsymbol{x} - \boldsymbol{y}|,$$

one has by (49) for all R large enough,

$$\int_{\boldsymbol{x} \in \Sigma_R^j, d^j(\boldsymbol{x}) < 1} \left| \frac{\partial v(\boldsymbol{x})}{\partial r} - \mathrm{i}kv(\boldsymbol{x}) \right|^2 ds(\boldsymbol{x}) \le C(1+R)^{-1},$$

where the constant C is independent of R. Combined with (43) with $\varepsilon = 1$ applied to $U^j(\varphi^j)$ and $U^{j+1}(\varphi^{j+1})$, this proves that

$$\lim_{R \to +\infty} \int_{\Sigma_R^j} \left| \frac{\partial v(\boldsymbol{x})}{\partial r} - \mathrm{i}kv(\boldsymbol{x}) \right|^2 ds(\boldsymbol{x}) = 0, \tag{50}$$

i.e., v satisfies the Sommerfeld radiation condition in a standard integrated form. Since $v \in H^1_{\mathrm{loc}}(\Omega^j \cap \Omega^{j+1})$ satisfies $\Delta v + k^2 v = 0$ in $\Omega^j \cap \Omega^{j+1}$, the Sommerfeld radiation condition (50), and vanishes on $\partial(\Omega^j \cap \Omega^{j+1}) \setminus \{S_j\}$, one can conclude that $v = 0$ by uniqueness of such problems in conical domains [21].

The second step of the proof is simpler. Arguing as in the proof of Proposition 2.7, it is enough to prove, in the case $g = 0$, that the function defined by $u = U^j(\varphi^j)$ in Ω^j, $j = 1, \dots, \mathcal{N}$, is equal to 0 everywhere, and to deduce from this that each $\varphi^j = 0$. Proceeding exactly as in that proof, one can show that $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega} \setminus P)$, where P is the set of corners of $\partial\Omega$, that u is a solution of (28) with $g = 0$, and that $\varphi^j = u|_{\Sigma^j}$ on $\Sigma^j \setminus P$. Moreover, thanks to the overlaps between the half-spaces Ω^j, one can deduce from Corollary 3.4 applied to each $U^j(\varphi^j)$ that u satisfies the Sommerfeld condition (2). This implies that $u = 0$ so that $\varphi^j = 0$ for all j. ∎

Theorem 3.5 is a (new) well-posedness result for (30) in the case of real k. And, indeed, numerical solution of (30) works well [6, 5] for real k. However, there are still

significant gaps in our theoretical understanding of this formulation. In particular, while (23) for complex wavenumber k can be written formally in operator form satisfying a Fredholm property, in the case when k is real we know of no function space setting for which this formulation makes sense, where the $D_{j,j+1}$ are well-defined bounded linear operators.

4. General configurations. Let us now recall how to extend the HSM formulation to the general problem presented in the introduction and extend the previous uniqueness result to this new formulation. More precisely, for a real wavenumber $k > 0$ we consider, as discussed in section 1.4, the isotropic Helmholtz equation (5) in a Lipschitz domain Ω, where $\rho - 1 \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$ are compactly supported and Ω is $\mathbb{R}^2$ or $\mathbb{R}^2$ minus a set of Lipschitz obstacles which are supposed to lie in a bounded domain. To ensure uniqueness we impose some constraint on ρ, e.g., that ρ is real-valued or that $\Im(\rho) \geq 0$. As announced in section 1.4, the problem we consider is to seek $u \in H^1_{\mathrm{loc}}(\Omega)$ that satisfies (5), the Sommerfeld radiation condition (2), and a boundary condition on $\Gamma = \partial\Omega$. To be specific, we will impose the homogeneous Neumann condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{51}$$

but the modifications to use other boundary conditions and/or include inhomogeneous terms are straightforward.

It is well known that this problem is uniquely solvable; in particular, with one of the above constraints on ρ, uniqueness follows by a Green's identity, the Rellich lemma, and unique continuation arguments (e.g., [15, Theorem 8.7]).

Let $\mathcal{O}$ be the interior of a convex polygon containing the supports of all these perturbations. The half-space matching method has been mainly presented for the case where $\mathcal{O}$ is a square (see [6]). But, whereas in the problem in section 3 the polygon $\mathcal{O}$ is given and the number of trace unknowns is imposed by the number of sides of $\mathcal{O}$, here we have freedom to choose the polygon $\mathcal{O}$. In particular, choosing $\mathcal{O}$ to be a triangle has the advantage of minimizing the number of trace unknowns. In what follows, we suppose that $\mathcal{O}$ is a polygon with $\mathcal{N}$ edges and we adopt the same notation as introduced in section 2 (see Figure 1) for the edges $\Gamma^1, \ldots, \Gamma^{\mathcal{N}}$, the angles between the edges, the $\mathcal{N}$ overlapping half-planes $\Omega^1, \ldots \Omega^{\mathcal{N}}$, their boundaries $\Sigma^1, \ldots, \Sigma^{\mathcal{N}}$, and their associated local coordinate systems.

Let us now introduce a bounded Lipschitz domain $\Omega_b \subset \Omega$ containing $\overline{\mathcal{O}} \cap \Omega$ such that $\partial\Omega_b \setminus \Gamma$ is connected (for examples, see Figure 4), and a partition $\Gamma_b^1, \ldots, \Gamma_b^{\mathcal{N}}$ of $\partial\Omega_b \setminus \Gamma$ such that, for all $j = 1, \ldots, \mathcal{N}$, Γ_b^j is connected and $\Gamma_b^j \subset \Omega^j$ with $\overline{\Gamma_b^j} \cap \Sigma^j = \emptyset$. There is, of course, no unique choice for such a partition, but the uniqueness result given in Theorem 4.1 below implies that the solution of the HSM formulation that we will write down is independent of the choice of the partition.

The unknowns of the HSM formulation are the traces $\varphi^1, \ldots, \varphi^{\mathcal{N}}$ of the solution u on the infinite lines $\Sigma^1, \ldots, \Sigma^{\mathcal{N}}$, and the restriction $u_b := u|_{\Omega^b}$ to the bounded domain Ω_b. Let us derive the equations linking $\varphi^1, \ldots, \varphi^{\mathcal{N}}$ and u_b. On the one hand, one can show as in the case of scattering by a polygon (see section 3.1) that the φ^j's satisfy the equations

$$\left| \begin{array}{ll} \varphi^j = D_{j-1,j}\, \varphi^{j-1} & \text{on } \Sigma^j \cap \Omega^{j-1}, \\ \varphi^j = D_{j+1,j}\, \varphi^{j+1} & \text{on } \Sigma^j \cap \Omega^{j+1}, \end{array} \right. \quad j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}, \tag{52}$$

where the operators $D_{j,j\pm1}$ are defined in (31). Compared to the problem of scattering by a polygon, the conditions for the traces on Γ^j (linked to the given data g in the

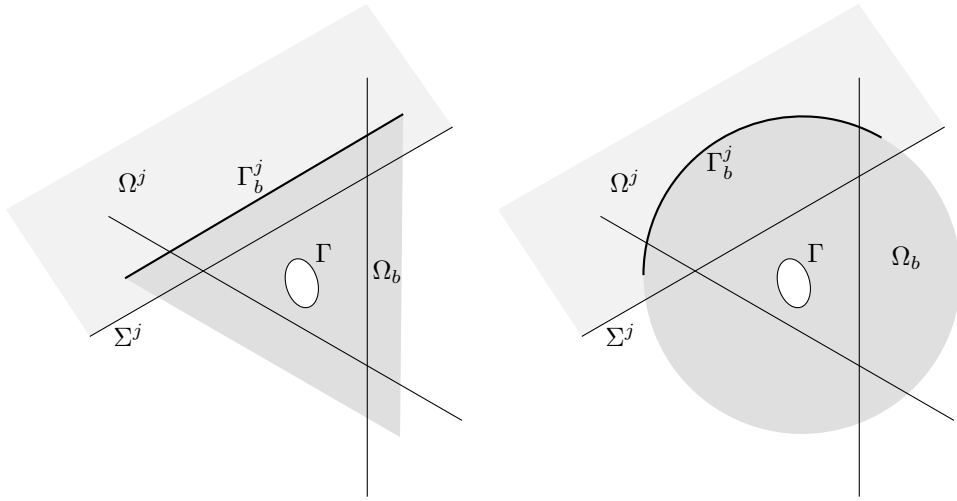

FIG. 4. *Two examples of possible geometries for our general configuration: $\mathcal{O}$ is the triangle bounded by the lines Σ^1, Σ^2, and Σ^3; Ω_b, shaded in darker gray, is a larger triangle in the left figure and a disc in the right figure.*

case of scattering by a polygon) have to be replaced by an equality between φ^j and u_b on Γ^j:

$$\varphi^j = u_b \quad \text{on } \Gamma^j,\ j = 1, \dots, \mathcal{N}.$$

As in the case of scattering by a polygon, since the wavenumber is real, the traces φ^j are not in L^2 but they decay like $|y_2^j|^{-1/2}$ at infinity and we will require, as in section 3.1, that, for $j = 1, \dots, \mathcal{N}$, φ^j satisfies the asymptotic condition (32) at $\pm\infty$.

On the other hand, we can derive a variational formulation for u_b in Ω_b. Since $-\Delta u_b - k^2 \rho u_b = f$ in Ω_b, f is supported in $\mathcal{O}' := \mathcal{O} \cap \Omega$, and u satisfies (51), the following Green's identity holds for all $v_b \in H^1(\Omega_b)$, where n is the normal pointing out of Ω_b:

$$\int_{\Omega_b} \left(\nabla u_b \cdot \overline{\nabla v_b} - k^2 \rho u_b \overline{v_b} \right) - \int_{\partial\Omega_b \setminus \Gamma} \frac{\partial u_b}{\partial n} \overline{v_b} = \int_{\mathcal{O}'} f \overline{v_b}. \tag{53}$$

The last idea is to replace the normal derivative on each part Γ_b^j of $\partial\Omega_b \setminus \Gamma$ by an integral representation as a function of φ^j. Indeed, since, for all $j = 1, \dots, \mathcal{N}$, $\Gamma_b^j \subset \Omega^j$, we must have

$$\frac{\partial u_b}{\partial n} - \mathrm{i}k u_b = \frac{\partial U^j(\varphi^j)}{\partial n} - \mathrm{i}k U^j(\varphi^j) \quad \text{on } \Gamma_b^j,\ j = 1, \dots, \mathcal{N}, \tag{54}$$

where $U^j(\varphi^j)$ is the restriction of the solution u to the half-plane Ω^j and is expressed in terms of φ^j in (11). Let us emphasize that our choice of imposing equality of Robin traces instead of normal derivatives is so that later we have uniqueness for the formulation for all $k > 0$. Note also that, since $\overline{\Gamma_b^j} \cap \Sigma^j = \emptyset$, the Robin trace of $U^j(\varphi^j)$ on Γ_b^j is C^∞ (see Lemma 2.3). To express (54) succinctly, we introduce the Dirichlet-to-Robin (DtR) operators Λ^j, given by

$$\Lambda^j \varphi := \left(\frac{\partial U^j(\varphi)}{\partial n} - \mathrm{i}k U^j(\varphi) \right) \Big|_{\Gamma_b^j}. \tag{55}$$

We note, by Lemma 2.3, that $\Lambda^j\varphi$ is well defined (at least) for all functions $\varphi \in L^2(\Sigma^j) + L^\infty(\Sigma^j)$. One can use (11) to write $\Lambda^j\varphi$ explicitly, for $j = 1, \dots, \mathcal{N}$, as

$$\Lambda^j\varphi(\boldsymbol{x}^j) = \int_{\mathbb{R}} \Big(\nabla\mathcal{H}(k; x_1^j - a^j, x_2^j - y_2^j) \cdot n(x_1^j, x_2^j) \\ -\mathrm{i}k\mathcal{H}(k; x_1^j - a^j, x_2^j - y_2^j)\Big)\, \varphi(a^j, y_2^j)\, \mathrm{d}y_2^j, \quad \boldsymbol{x}^j = (x_1^j, x_2^j) \in \Gamma_b^j.$$

With this notation, the equality (54) can be written as

$$\frac{\partial u_b}{\partial n} - \mathrm{i}ku_b = \Lambda^j\varphi^j \quad \text{on } \Gamma_b^j,\ j = 1, \dots, \mathcal{N}. \tag{56}$$

Our complete formulation reads as follows:

$$\left|\begin{array}{ll} \varphi^j = D_{j-1,j}\,\varphi^{j-1} & \text{on } \Sigma^j \cap \Omega^{j-1}, \\ \varphi^j = u_b & \text{on } \Gamma^j, \\ \varphi^j = D_{j+1,j}\,\varphi^{j+1} & \text{on } \Sigma^j \cap \Omega^{j+1}, \end{array}\right. \qquad j \in \mathbb{Z}/\mathcal{N}\mathbb{Z},$$

$$\forall v_b \in H^1(\Omega_b), \\ \int_{\Omega_b} \big(\nabla u_b \cdot \overline{\nabla v_b} - k^2\rho u_b\overline{v_b}\big) - \mathrm{i}k\sum_{j=0}^{3}\int_{\Gamma_b^j} u_b\overline{v_b} - \sum_{j=0}^{3}\int_{\Gamma_b^j} \Lambda^j\varphi^j\overline{v_b} = \int_{\mathcal{O}'} f\overline{v_b}, \tag{57}$$

where the operators $D_{j,j\pm1}$ are defined in (31), and the operators Λ^j in (55).

The following theorem gives the well-posedness of this formulation.

THEOREM 4.1. *Suppose $k > 0$. There exists, for each $f \in L^2(\mathcal{O}')$, a unique solution $(\varphi^1, \dots, \varphi^{\mathcal{N}}, u_b) \in L^2_{\mathrm{loc}}(\Sigma^1) \times \cdots \times L^2_{\mathrm{loc}}(\Sigma^{\mathcal{N}}) \times H^1(\Omega_b)$ of* (57) *such that, for all $j = 1, \dots, \mathcal{N}$, φ^j satisfies the radiation condition* (32).

Existence holds by the construction above of a solution to (57) from the unique solution of (5), (2), (51). Uniqueness follows from the following proposition since (5), (2), (51) is well-posed.

PROPOSITION 4.2. *Suppose $k > 0$, and let $(\varphi^1, \dots, \varphi^{\mathcal{N}}, u_b) \in L^2_{\mathrm{loc}}(\Sigma^1) \times \cdots \times L^2_{\mathrm{loc}}(\Sigma^{\mathcal{N}}) \times H^1(\Omega_b)$ be a solution of* (57) *(with $f \in L^2(\mathcal{O}')$) such that, for all $j = 1, \dots, \mathcal{N}$, φ^j satisfies the radiation condition* (32). *Then, for all $j \in \mathbb{Z}/\mathcal{N}\mathbb{Z}$, the compatibility equations* (22) *hold and $u_b = U^j(\varphi^j)$ in $\Omega_b \cap \Omega^j$. Moreover, the function defined by $u = U^j(\varphi^j)$ in Ω^j, for $j = 1, \dots, \mathcal{N}$, and by $u = u_b$ in Ω_b is the unique solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of* (5), (2), (51). *Further, $\varphi^j = u|_{\Sigma^j}$ for each j.*

Proof. Let $(\varphi^1, \dots, \varphi^{\mathcal{N}}, u_b) \in L^2_{\mathrm{loc}}(\Sigma^1) \times \cdots \times L^2_{\mathrm{loc}}(\Sigma^{\mathcal{N}}) \times H^1(\Omega_b)$ be a solution of (57) such that, for all $j = 1, \dots, \mathcal{N}$, φ^j satisfies the radiation condition (32), and let us denote by $u_\infty \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\mathcal{O}})$ the unique solution of (28), (2) with $g = u_b|_{\partial\mathcal{O}} \in H^{1/2}(\partial\mathcal{O})$. Then, as shown in section 3, the set of traces of u_∞ on $\Sigma^1, \dots, \Sigma^{\mathcal{N}}$ solves (30), i.e., the first three equations of (57). It follows from Theorem 3.5 and from (57) that $\varphi^j = u_\infty|_{\Sigma^j}$ and $u_\infty = U^j(\varphi^j)$ in Ω^j for $j = 1, \dots, \mathcal{N}$. In particular, the compatibility equations (22) hold and $\varphi^j = u_\infty = u_b$ on Γ^j for $j = 1, \dots, \mathcal{N}$. Moreover, we deduce from the last equation of (57) that $-\Delta u_b - k^2\rho\, u_b = f$ in Ω_b, that

$$\frac{\partial u_b}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$

and that (56) holds. By definition (55) of the DtR operators, we then have

$$\Lambda^j\varphi^j = \left(\frac{\partial u_\infty}{\partial n} - \mathrm{i}ku_\infty\right)\Big|_{\Gamma_b^j}, \quad j = 1, \dots, \mathcal{N}. \tag{58}$$

Consequently, $v := u_b - u_\infty$ belongs to $H^1(\Omega_b \backslash \overline{\mathcal{O}})$ and satisfies

$$\left| \begin{array}{lll} \Delta v + k^2 v = 0 & \text{in} & \Omega_b \backslash \overline{\mathcal{O}}, \\ v = 0 & \text{on} & \partial \mathcal{O}, \\ \dfrac{\partial v}{\partial n} - \mathrm{i}kv = 0 & \text{on} & \partial \Omega_b \backslash \Gamma, \end{array} \right. \tag{59}$$

the last of these equations following from (56) and (58). But, for every $k > 0$, this homogeneous problem has no solution except $v = 0$. (To see this apply Green's identity (cf. (53)) in $\Omega_b \backslash \overline{\mathcal{O}}$ to deduce that $\int_{\partial\Omega_b \backslash \Gamma} |v|^2 = 0$, so that $v = \partial v / \partial n = 0$ on $\partial \Omega_b \setminus \Gamma$, and apply Holmgren's uniqueness theorem [11, p. 104].) Thus $u_b = u_\infty$ in $\Omega_b \backslash \overline{\mathcal{O}}$ (in particular, $u_b = u_\infty = U^j(\varphi^j)$ in $\Omega_b \cap \Omega^j$, $j = 1, \ldots, \mathcal{N}$) so that the function

$$w := \begin{cases} u_b & \text{in} \quad \Omega_b, \\ u_\infty & \text{in} \quad \mathbb{R}^2 \backslash \overline{\mathcal{O}} \end{cases}$$

is well defined, and is the unique solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of (5), (2), (51). ∎

5. Conclusion. The objective of this paper was to prove the well-posedness of the HSM formulation applied to the Helmholtz equation (5) in the case of a real wavenumber $k > 0$. This objective has been achieved: we have proved well-posedness provided we impose the radiation condition (32) at infinity on the trace unknowns φ^j of the HSM equations, this radiation condition analogous to the standard Sommerfeld radiation condition for the original boundary value problem. Let us recall that the results of the present paper also complete a proof of well-posedness of the complex-scaled HSM formulation presented in [4, section 5].

An open question is whether the radiation condition (32) on the trace unknowns, while natural, is necessary for uniqueness. The answer is not at all clear to us. But one piece of evidence that suggests that this radiation condition may not, in fact, be needed is that one achieves accurate results in numerical experiments by simply truncating the trace unknowns, setting, for each j, $\varphi^j = 0$ outside some sufficiently large finite section of Σ^j, not making any use of the radiation condition (32) (see [6, Fig. 7]).

Let us finish by mentioning that our main results should be extendable to more complex configurations with obstacles extending to infinity. For instance, one might consider the general configuration of section 4 but add a screen that is a semi-infinite line γ^1, choosing the half-planes Ω^j so that γ^1 is orthogonal to Σ^1 and $\gamma^1 \cap \Omega^j = \emptyset$, for all $j \neq 1$ (see the left-hand side of Figure 5). In this case $\Omega^1 \setminus \gamma^1$ has two connected components that are quarter-planes. If γ^1 is sound soft or sound hard, i.e., the solution satisfies a Dirichlet or Neumann homogeneous boundary condition on γ^1, one can derive an expression for $U^1(\varphi^1)$, the solution in Ω^1 given Dirichlet data φ^1 on Σ^1, by solving separately in each quarter-plane, combining formula (13) with reflection arguments.

More interesting is the case where two parallel semi-infinite lines $\gamma^j_\pm$ are introduced, so that now, for some j, Ω^j has three connected components, two of them quarter-planes and one of them a semi-infinite strip which is a waveguide (this is the case for Ω^1 and Ω^3 in the right-hand side of Figure 5). One can derive an expression for $U^j(\varphi^j)$ in each quarter-plane as above, and an expression in the waveguide by solving as a modal series expansion.

For each of these configurations, and for many other variations on these geometries, we can write down HSM formulations, and we expect that one should be able to

prove well-posedness of the HSM formulation and equivalence with other formulations by adapting the results proved in the present paper.

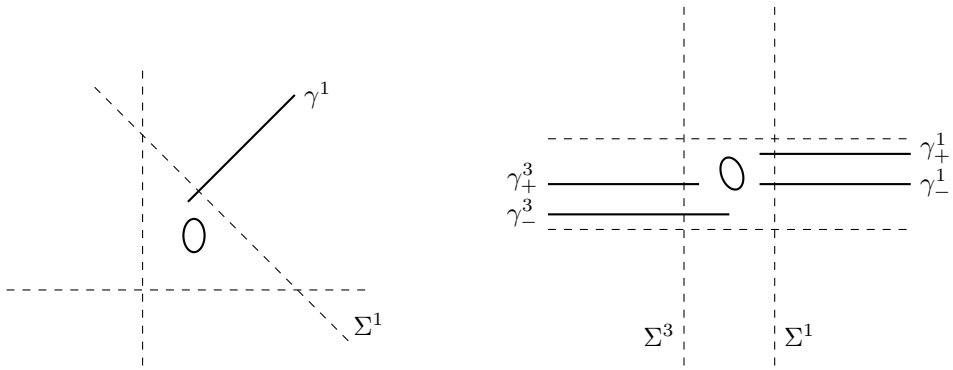

FIG. 5. *Two geometries that could be considered.*

Note that if only one semi-infinite line is introduced, as in the left-hand side of Figure 5, and no other obstacles or inhomogeneities are present, one recovers the famous Sommerfeld half-plane problem [29, section 38] that can be solved using the Wiener–Hopf method (or other techniques). Analytical methods, notably the Wiener–Hopf method, have been extended to a variety of more complex configurations with waveguides and wedges, e.g., [26, 23, 22], though not to all configurations that should be treatable by the extensions of the HSM method to unbounded obstacles that we suggest above.

An interesting question is whether our uniqueness and well-posedness results for the HSM formulation can be extended to open waveguides. (The HSM method has been used to compute numerical solutions by Ott [27] in such cases where the medium in each half-space Ω^j is stratified in the x_2^j direction.) The difficulty is that the corresponding Green's function is no longer available in closed form, which makes the study of the properties of the half-plane solution more intricate. A further difference with the cases studied in this paper, and the extensions suggested above, is that, due to the possible existence of guided waves in the open waveguides, the traces φ^j may have an oscillatory nondecaying behavior at infinity that should be taken into account in a modified radiation condition, replacing (32).

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