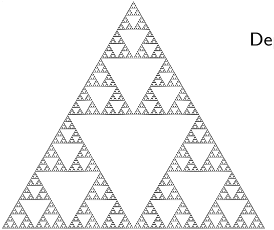
PDE and Integral Equation Formulations for Scattering by Fractal Screens

Simon Chandler-Wilde



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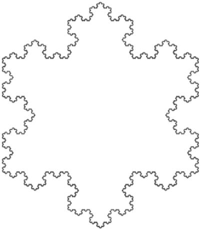


Joint work with: Dave Hewett (UCL) and Andrea Moiola (Reading)

Waves 2017, Minnesota

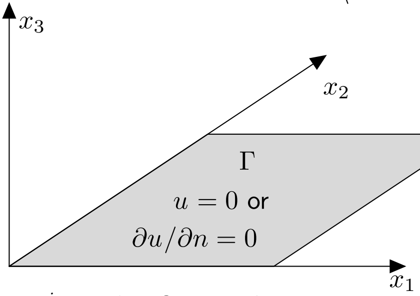
May 17th, International Day Against Homophobia, Biphobia and Transphobia 

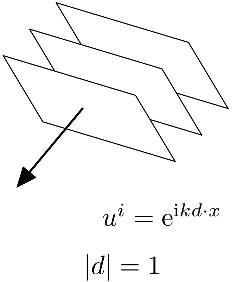
Acoustic scattering by planar screens

Γ a bounded subset of $\Gamma_\infty := \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong \mathbb{R}^n$, $n = 1, 2$

$$(\Delta + k^2)u = 0$$

in $D := \mathbb{R}^{n+1} \setminus \overline{\Gamma}$





$u^s := u - u^i$ satisfies Sommerfeld Radiation Condition (SRC) at infinity

$\partial u^s/\partial r - iku^s = o\left(r^{-n/2}\right)$ uniformly as $r = |x| \to \infty$.

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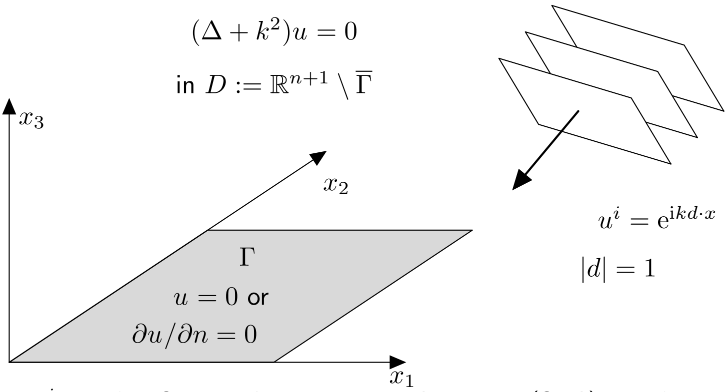

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Classical problem when Γ is **Lipschitz open set or smoother**.

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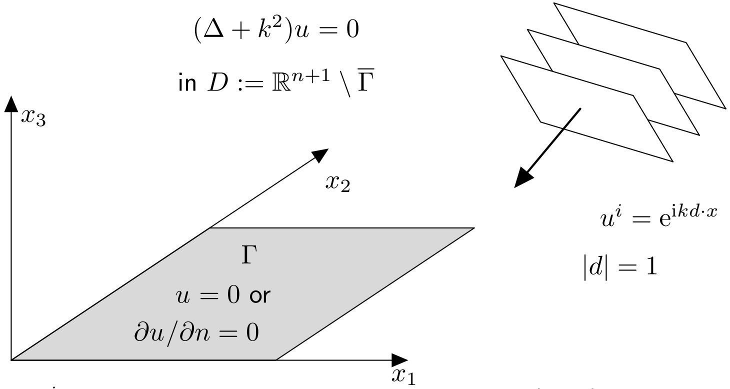

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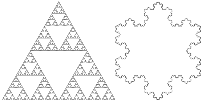
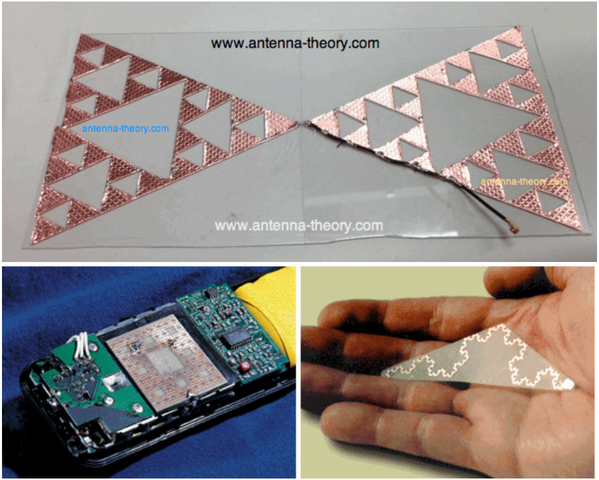


What about **rougher** Γ, e.g. **fractal** or with **fractal boundary**?

Fractal antennas

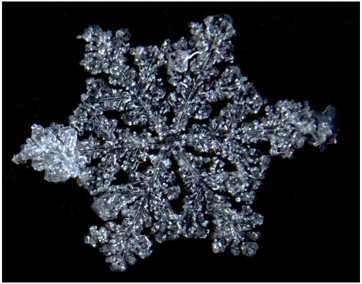


(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

Attractive because of wideband/multiband performance

Not yet analysed by mathematicians

Other applications



Scattering by ice crystals in atmospheric physics
- e.g. Westbrook and Nicol (2015) -
Meteorology at University of Reading

Fractal apertures in optics
- e.g. Huang, Christian, McDonald (2017)

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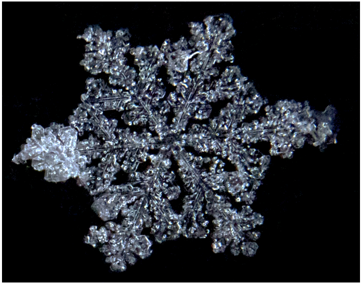

Fractal apertures in optics
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These are all examples of **'diffractals'** (Berry 1979), **waves encountering fractals**.

Scattering by apertures in infinite planar screens

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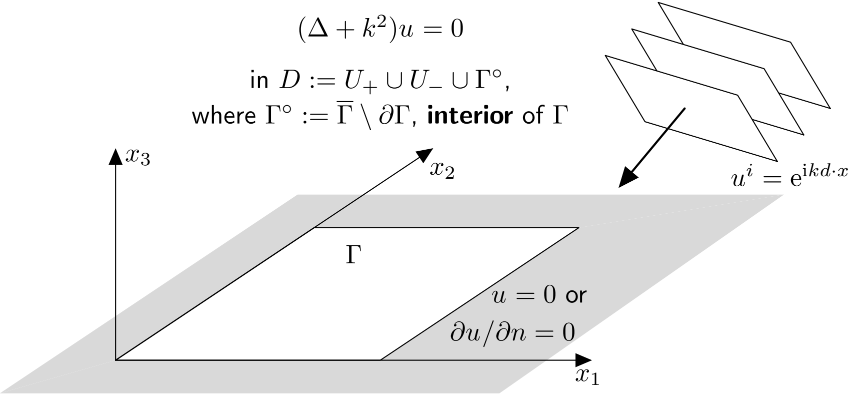
$$(\Delta + k^2)u = 0$$

in $D := U_+ \cup U_- \cup \Gamma^\circ$,
where $\Gamma^\circ := \overline{\Gamma} \setminus \partial\Gamma$, **interior** of Γ



The Sommerfeld radiation condition is satisfied by:

- u in the **lower** half-space, U_-
- $u^s := u - u^i - u^r$ (u^r a reflected plane wave) in the **upper** half-space, U_+

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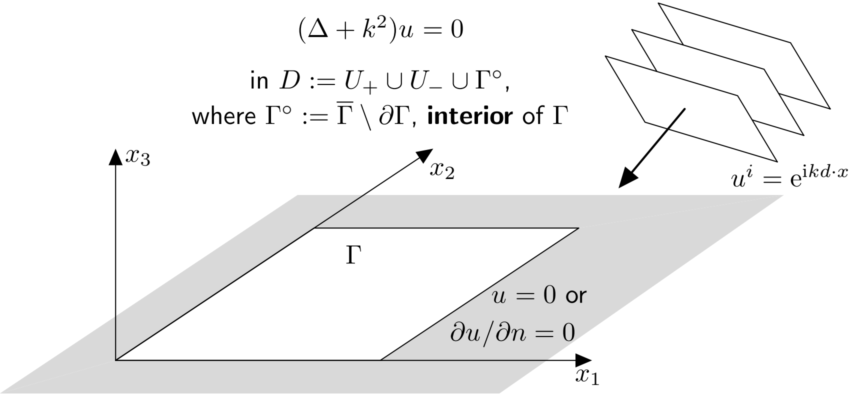

Screen and aperture problems classically connected by **Babinet's principle**:

scattered field for screen = **scattered field** for aperture,

in **upper** half-space, but for **opposite boundary conditions**.

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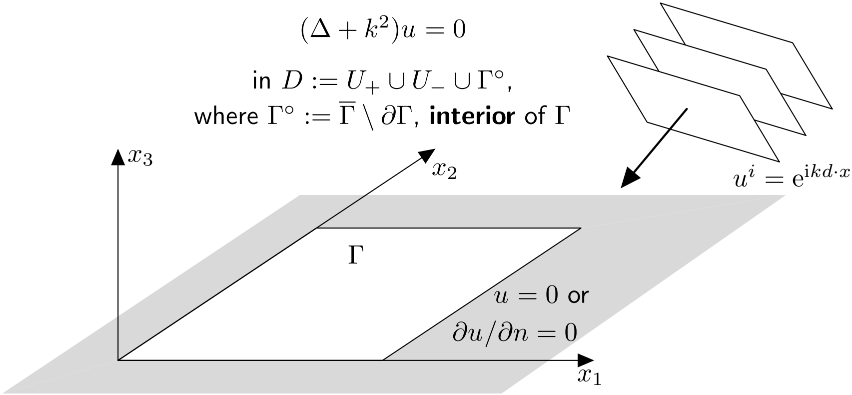

Screen and aperture problems classically connected by **Babinet's principle**:

scattered field for screen = −**transmitted field** for aperture,

in **lower** half-space, again for **opposite boundary conditions**.

Overview of Talk

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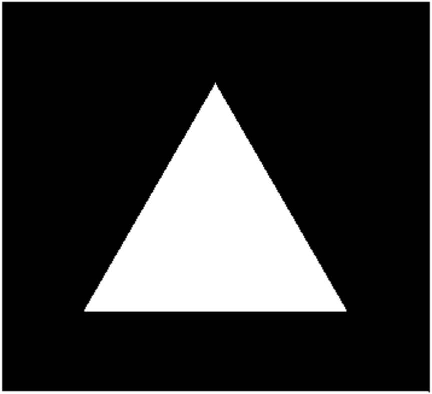
2. Warm up
   - **Examples/questions** to get us thinking
   - The **main questions** – look ahead to answers
3. **PDE and BIE formulations**
   - for **regular** screens
   - for **rough** screens, e.g. **fractal** or **fractal boundary**
4. **Convergence** of regular screens to irregular, prefractals to fractals?
5. Recap, references & many **open problems**

Example 1: Fractal aperture in sound hard screen



Aperture in infinite sound hard $\left(\frac{\partial u}{\partial n} = 0\right)$ screen: Area $= 1$

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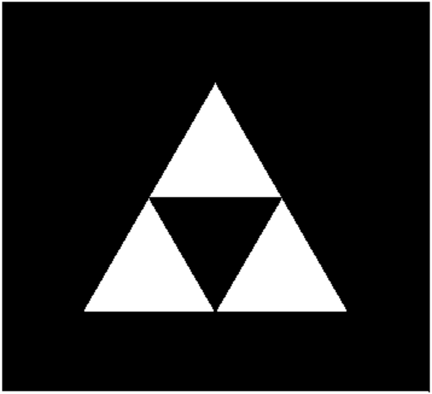

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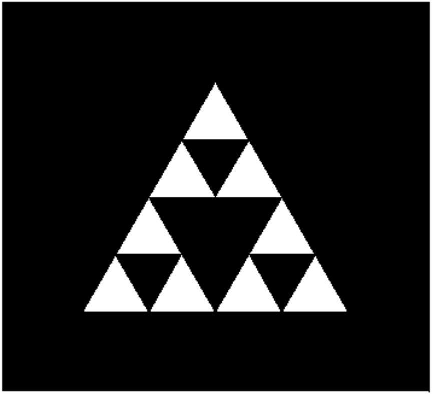

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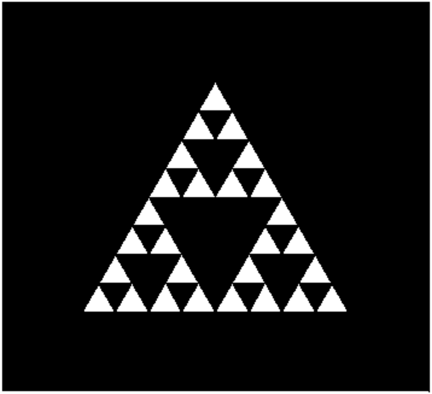

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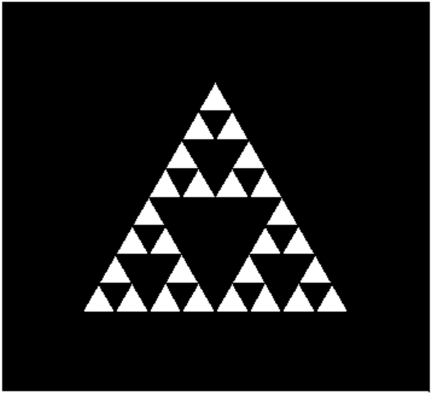
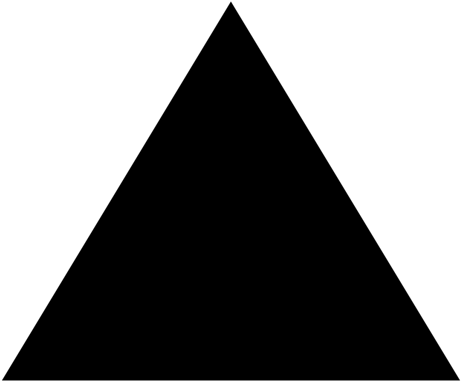


Aperture in infinite sound hard $\left(\frac{\partial u}{\partial n} = 0\right)$ screen: Area $= (3/4)^3$

Question: Is the transmitted field **zero** or **non-zero** in the limit? (The limiting aperture is the **Sierpinski triangle fractal** with zero area.)

Example 2: Sierpinski triangle screen



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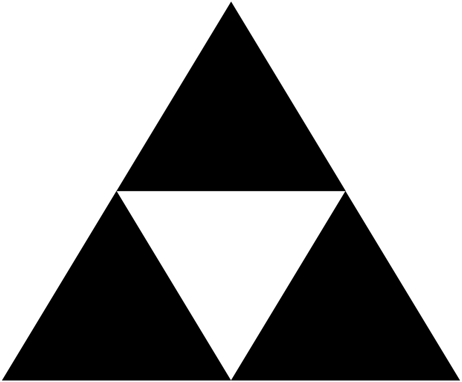

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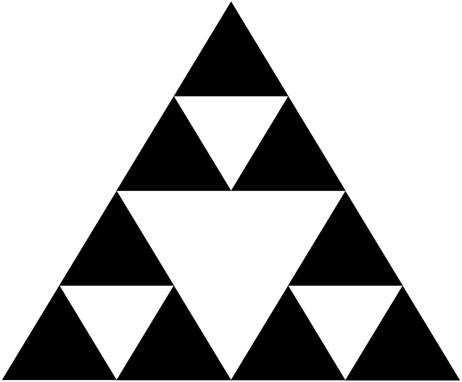

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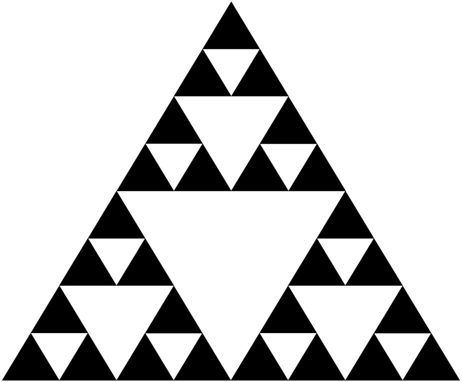

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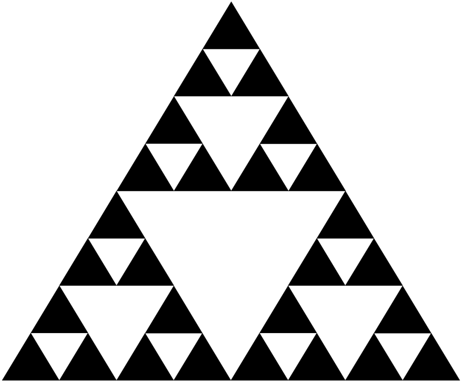

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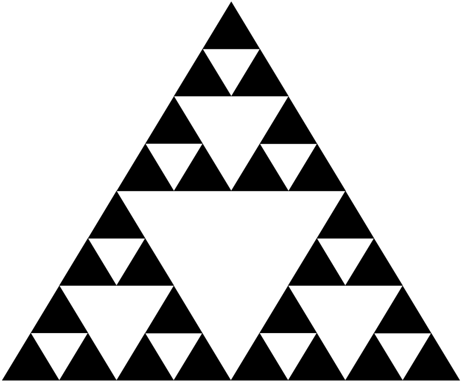

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By **Babinet's principle** this is the same question as on the previous slide.

Example 3: Scattering by Cantor Dust

Let the closed set $C_\alpha \subset [0,1]$ denote the standard Cantor set $(0 < \alpha < 1/2)$

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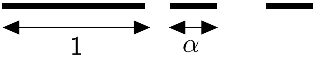

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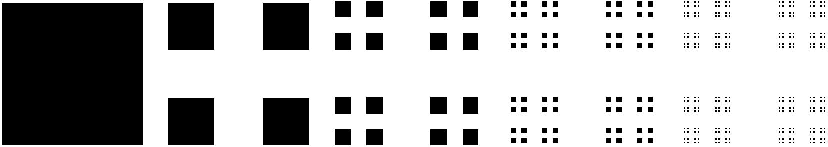

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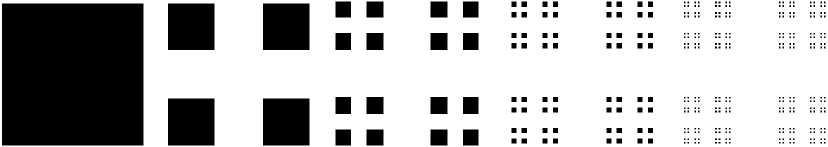

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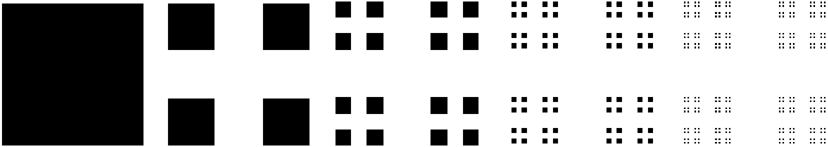

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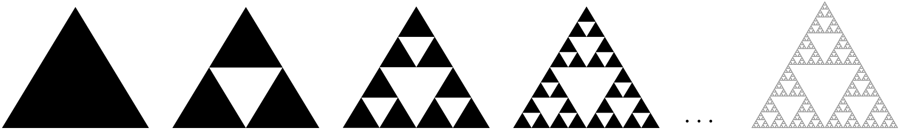
Question: Is the scattered field **zero** or **non-zero** for the sound-soft scattering problem with $\Gamma = C_\alpha^2$?

Scattering by fractal (and other complicated) screens/apertures



Lots of interesting mathematical/computational questions:

- Can we **formulate** well-posed BVPs and equivalent BIEs?

  .
- Do prefractal solutions **converge** to fractal solutions?

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- Are there algorithms to **compute** the scattered field?

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- If the screen/aperture has **empty interior**, does it scatter?
- Does fractal dimension play a role?

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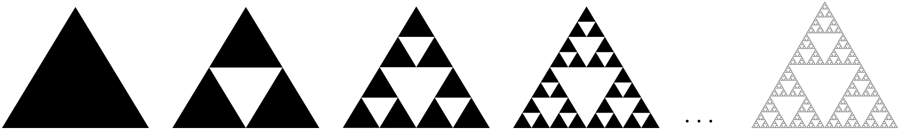

Lots of interesting mathematical/computational questions:

- Can we **formulate** well-posed BVPs and equivalent BIEs?
  .**Yes – in fact infinitely many.**
- Do prefractal solutions **converge** to fractal solutions?
  .**Yes, and this helps us select which fractal solution is physical.**
- Are there algorithms to **compute** the scattered field?
  .**Yes, but this work in progress.**
- If the screen/aperture has **empty interior**, does it scatter? **It depends.**
- Does fractal dimension play a role? **Very much.**

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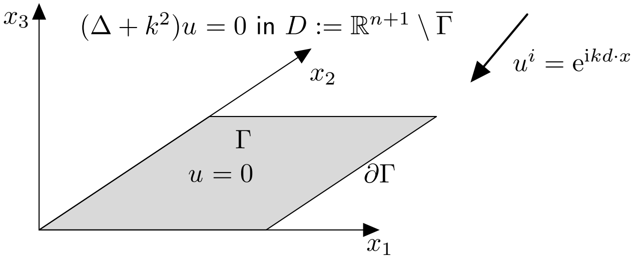
Formulations for Regular Screens (Sound Soft Case)

BVP-C: Classical BVP Formulation (Bouwkamp, “Diffraction Theory”, 1954)



- Implicit assumption that $u \in C^2(D)$, indeed is smooth up to the boundary except on $\partial\Gamma$
- $u^s := u - u^i$ satisfies Sommerfeld radiation condition (SRC)
- Behaviour near $\partial\Gamma$ controlled by “edge conditions”, notably (Meixner 1949) $\int_G \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x < \infty$ for every bounded $G \subset D$

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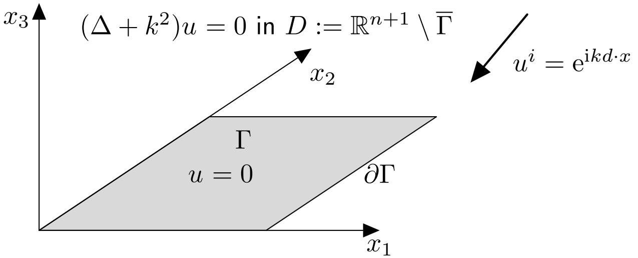

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Theorem (Meixner '49, Levine '64, Wolfe '69, Stephan '87, C-W, Hewett 2016)

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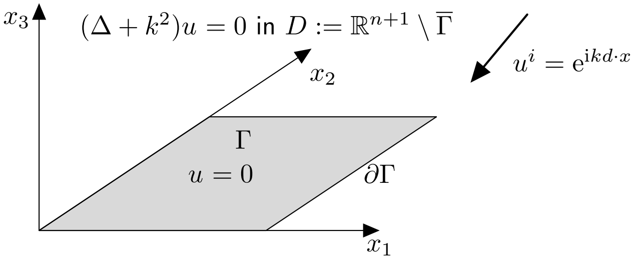

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Sobolev spaces on $\Gamma \subset \Gamma_\infty = \mathbb{R}^n$

For $s \in \mathbb{R}$ let $H^s(\Gamma_\infty) = H^s(\mathbb{R}^n) = \{u \in \mathcal{S}^*(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, \mathrm{d}\xi < \infty\}$

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$$H^s(\Omega) := \{u|_\Omega : u \in H^s(\Gamma_\infty)\}$$ **RESTRICTION**

$$\tilde{H}^s(\Omega) := \overline{C_0^\infty(\Omega)}^{H^s(\Gamma_\infty)}$$ **CLOSURE**

$$H_F^s := \{u \in H^s(\Gamma_\infty) : \operatorname{supp} u \subset F\}$$ **SUPPORT**

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Further

$$\tilde{H}^{\pm 1/2}(\Gamma^\circ) \subset H^{\pm 1/2}_{\overline{\Gamma}}$$

with **equality** if Γ is open and C^0.

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Further

$$\tilde{H}^{\pm 1/2}(\Gamma^\circ) \subset H^{\pm 1/2}_{\overline{\Gamma}}$$

with **equality** if Γ is open and C^0.

But **equality does not hold in general and this is key!**

Sobolev spaces on $\Gamma \subset \Gamma_\infty = \mathbb{R}^n$

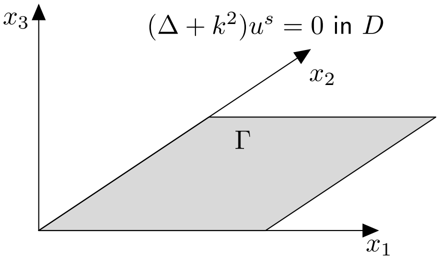
Where $U_+, U_- \subset \mathbb{R}^{n+1}$ are the half-spaces **above** and **below** Γ_∞, define standard **trace** operators

$$\gamma_\pm : W^1(U_\pm) \to H^{1/2}(\Gamma_\infty)$$

by $\gamma_\pm u = u|_{\Gamma_\infty}$, for $u \in W^1(U_\pm) \cap C(\overline{U_\pm})$.

Formulations for Regular Screens (Sound Soft Case)

BVP-S: Sobolev space formulation (Stephan 1987)



Require:

- $u^s \in C^2(D) \cap W^{1,loc}(D)$
- u^s satisfies SRC
- $\gamma_\pm u^s|_{\Gamma^\circ} = -u^i|_{\Gamma^\circ} \in H^{1/2}(\Gamma^\circ)$

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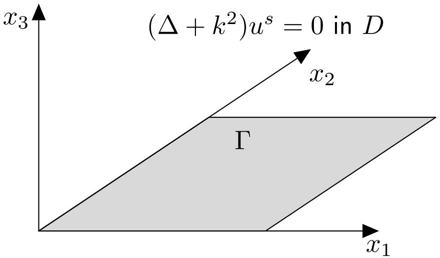

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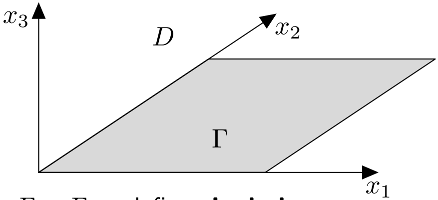
Theorem (Stephan 1987, C-W, Hewett 2016)

This formulation is **equivalent** *to the classical formulation* **FC**, *and both are uniquely solvable if Γ is a C^0 open set.*

Formulations for Regular Screens (Sound Soft Case)

BIE formulation (Stephan 1987)



Given a bounded open $\Gamma \subset \Gamma_\infty$, define **single-layer** operators $\mathcal{S} : \tilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^{1,loc}(D)$ and $S : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ by

$$\mathcal{S}\phi(x) = \int_\Gamma \Phi(x,y)\phi(y)\,\mathrm{d}s(y), \quad x \in D \quad \text{and} \quad S\phi = \left.(\gamma_\pm \mathcal{S}\phi)\right|_\Gamma .$$

$$\Phi(x,y) := \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} \quad \text{in 3D.}$$

Formulations for Regular Screens (Sound Soft Case)

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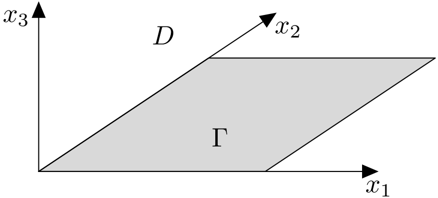

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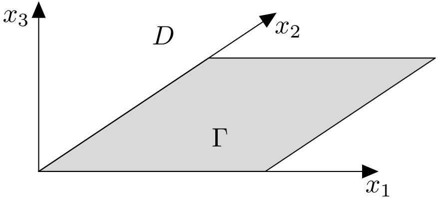

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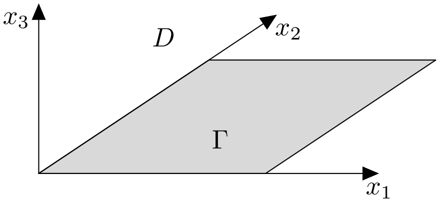

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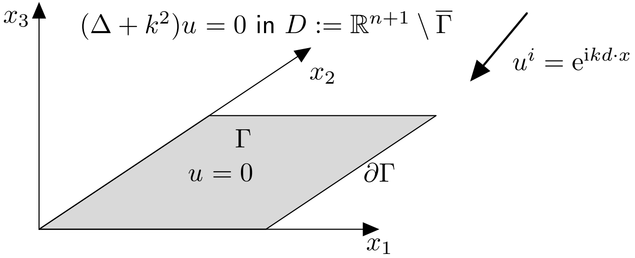
Theorem (Stephan 1987, C-W & Hewett 2015)

This BIE formulation has exactly one solution **for every open** Γ, *and this solution satisfies* **BVP-S** *and* **BVP-C**. *Further,* $\psi = -[\partial_n u^s]$.

Formulations for Regular Screens (Sound Soft Case)

BVP-W: Weak BVP Formulation



Require:

- $u \in C^2(D) \cap W_0^{1,loc}(D)$, where $W_0^1(D)$ is closure of $C_0^\infty(D)$ in $W^1(D)$
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Formulations for Regular Screens (Sound Soft Case)

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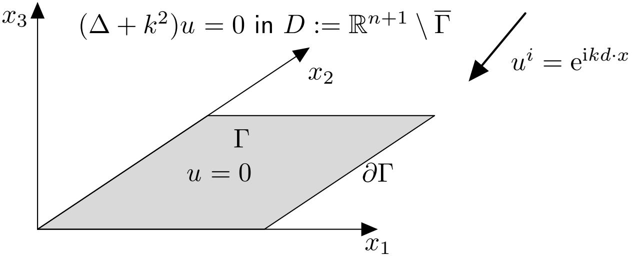

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Theorem

This formulation has a unique solution **for every bounded** Γ.

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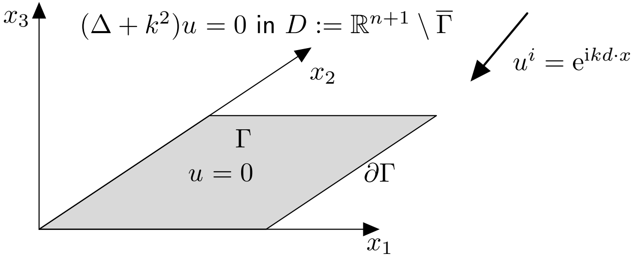

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Theorem

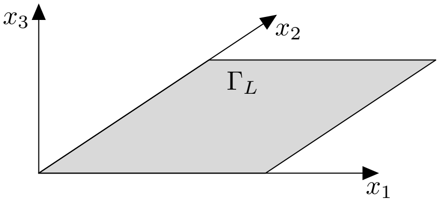
This formulation has a unique solution **for every bounded** Γ.

Not the end of the story! This solution is **may or may not** be the **right solution** and **may or may not** be the same as the solution of the **BIE**.
(Though all formulations have the same unique solution if Γ is C^0 open set.)

Overview of Talk

1. The **screen/aperture problems** and applications
2. Warm up
   - **Examples/questions** to get us thinking
   - The **main questions** – look ahead to answers
3. **PDE and BIE formulations**
   - for **regular** screens
   - for **rough** screens, e.g. **fractal** or **fractal boundary**
4. **Convergence** of regular screens to irregular, prefractals to fractals?
5. Recap, references & many **open problems**

BIE for General Bounded Sound Soft Screen $\Gamma \subset \mathbb{R}^n$



Start with **Lipschitz open** $\Gamma_L \subset \Gamma_\infty$. Define **single-layer** operators

$$\mathcal{S}_L\phi(x) = \int_{\Gamma_L} \Phi(x,y)\phi(y)\,\mathrm{d}s(y), \quad \text{and} \quad S_L\phi = (\gamma_\pm \mathcal{S}\phi)|_{\Gamma_L}.$$

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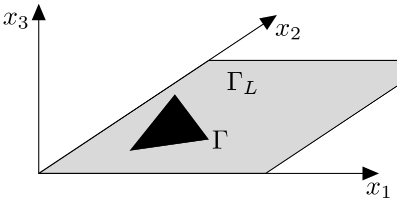

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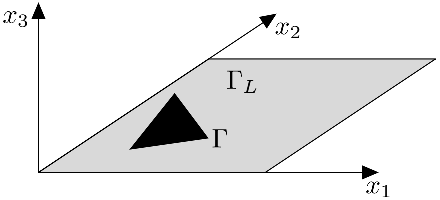

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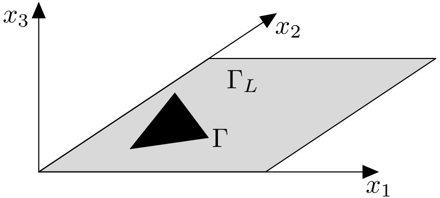

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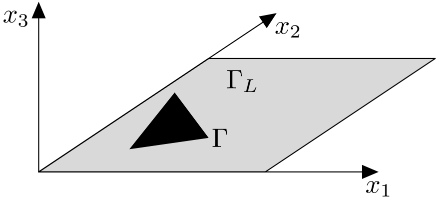

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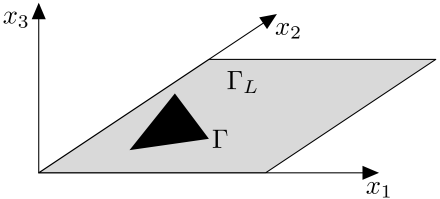

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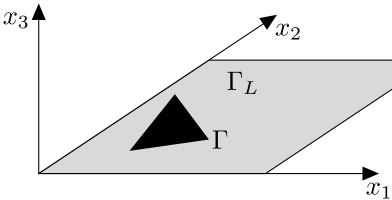

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Let $\langle\cdot,\cdot\rangle$ be the standard **duality pairing** on $H^{1/2}(\Gamma_L) \times \tilde{H}^{-1/2}(\Gamma_L)$:
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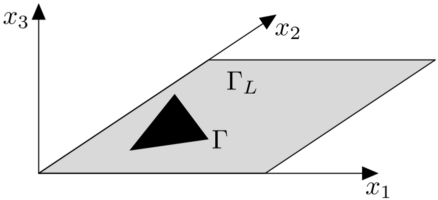

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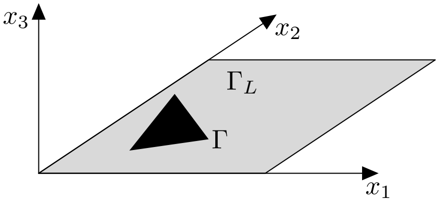

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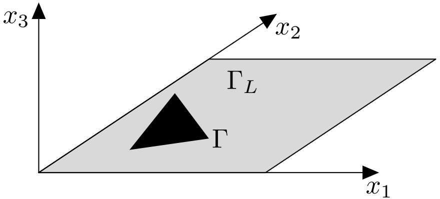

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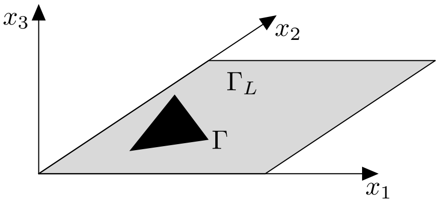
Theorem (Ha Duong 1992, C-W & Hewett 2015)

For some $c > 0$, $|\langle S_L\phi, \phi\rangle| \geq c\|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma_L)}$, *for* $\phi \in \tilde{H}^{-1/2}(\Gamma_L)$

BIE for General Bounded Sound Soft Screen $\Gamma \subset \mathbb{R}^n$



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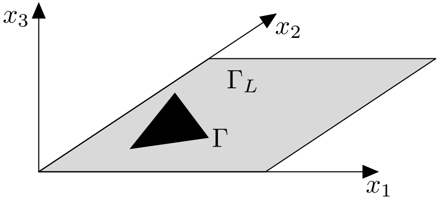
Case 1: $\tilde{H}^{-1/2}(\Gamma^\circ) = H^{-1/2}_{\overline{\Gamma}}$, e.g. Γ is open and C^0.
BIE-V has exactly one solution by Lax-Milgram.

BIE for General Bounded Sound Soft Screen $\Gamma \subset \mathbb{R}^n$



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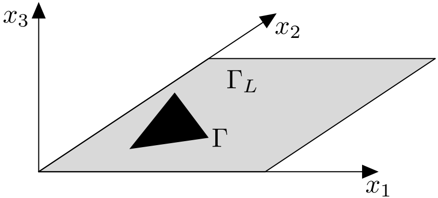
Case 1a: $\Gamma^\circ = \emptyset$ and $\{0\} = \tilde{H}^{-1/2}(\Gamma^\circ) = H^{-1/2}_{\overline{\Gamma}}$, e.g. $\overline{\Gamma}$ is countable or $\dim_H(\overline{\Gamma}) < n-1$. **BIE-V** has only the zero solution, $u^s = 0$.

BIE for General Bounded Sound Soft Screen $\Gamma \subset \mathbb{R}^n$



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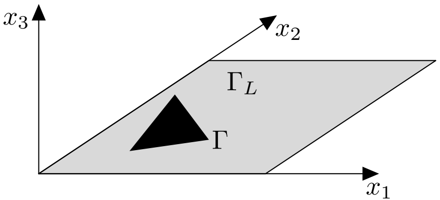
Case 2: $\tilde{H}^{-1/2}(\Gamma^\circ) \subsetneq H^{-1/2}_{\overline{\Gamma}}$.
BIE-V has infinitely many solutions.

BIE for General Bounded Sound Soft Screen $\Gamma \subset \mathbb{R}^n$



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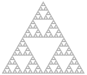
Case 2a: $\Gamma^\circ = \emptyset$ and $\{0\} = \tilde{H}^{-1/2}(\Gamma^\circ) \subsetneq H^{-1/2}_{\overline{\Gamma}}$, e.g. $\dim_H(\overline{\Gamma}) > n-1$, e.g.

$\Gamma =$ . **BIE-V** has infinitely many solutions.

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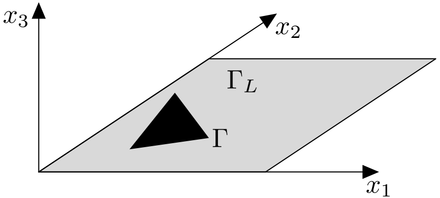

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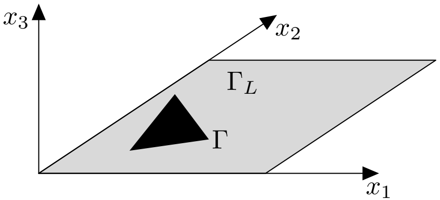

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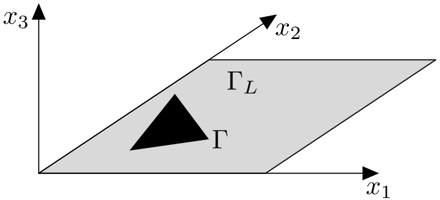

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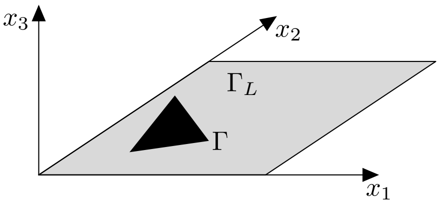

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Infinitely many ($\aleph_0$) choices; distinct choices have **distinct solutions** for a.e. d.

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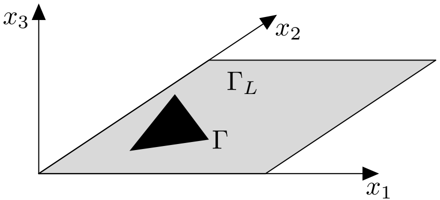

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But are any of these infinitely many solutions physical?

Overview of Talk

1. The **screen/aperture problems** and applications
2. Warm up
   - **Examples/questions** to get us thinking
   - The **main questions** – look ahead to answers
3. **PDE and BIE formulations**
   - for **regular** screens
   - for **rough** screens, e.g. **fractal** or **fractal boundary**
4. **Convergence** of regular screens to irregular, prefractals to fractals?
5. Recap, references & many **open problems**

Prefractal convergence from above

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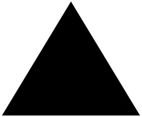
$\Gamma_1 =$ 

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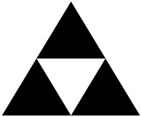
$\Gamma_2 =$ 

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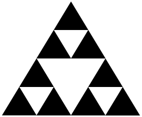
$\Gamma_3 =$ 

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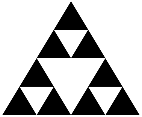
$\Gamma_3 =$ 

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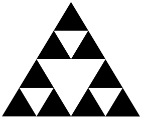
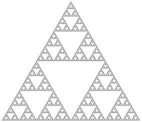
$\Gamma_3 =$  $\Gamma =$ 

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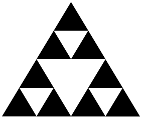
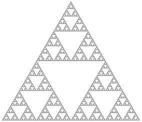
$\Gamma_3 =$  $\Gamma =$ 

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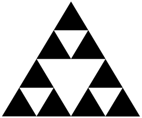
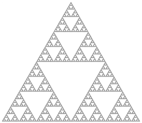
$\Gamma_3 =$  $\Gamma =$ 

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This solution **coincides with the solution to BVP-w**
in which the boundary condition is enforced by $u \in W_0^{1,loc}(D)$.

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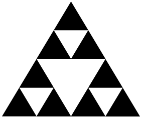
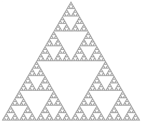
$\Gamma_3 =$  $\Gamma =$ 

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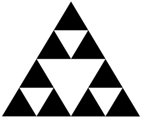
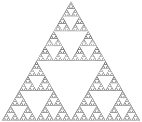
$\Gamma_3 =$  $\Gamma =$  $\dim_H \Gamma = \log_2 3 > n - 1.$

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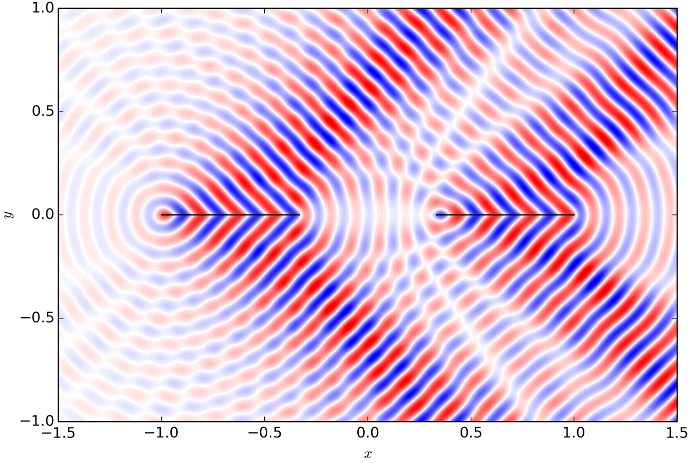
Accurate spectral computations by Mikaël Slevinsky (Slevinsky, Olver 2017).



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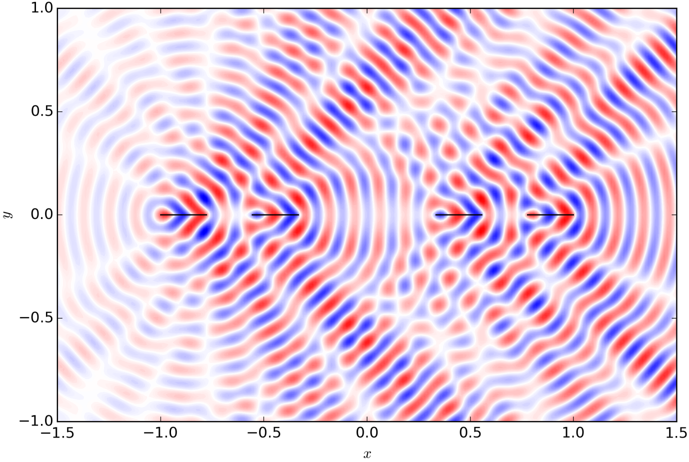

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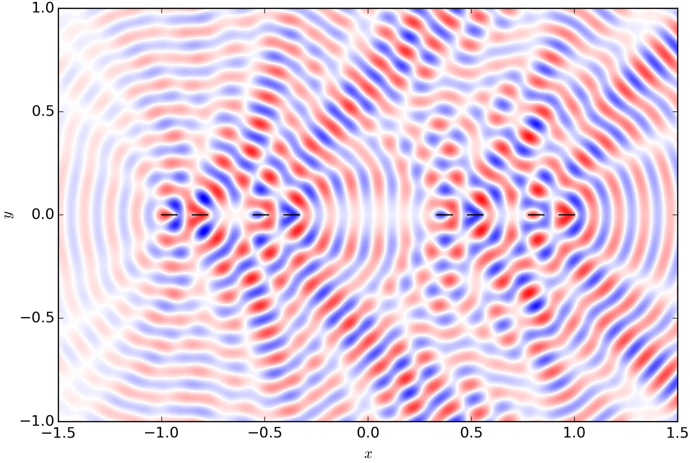

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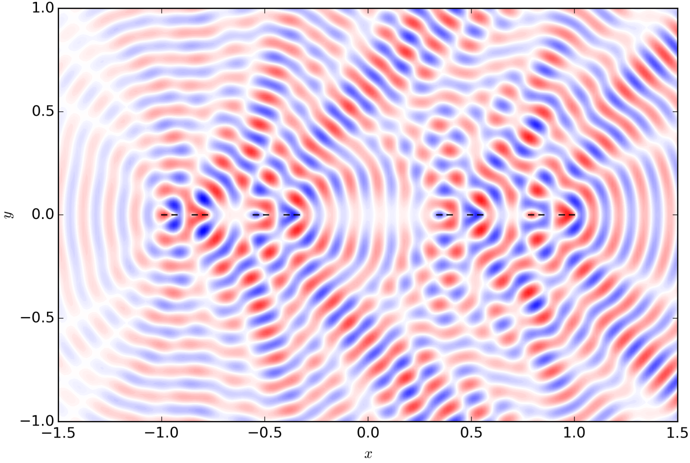

Γ_4 and $\mathrm{Re}\, u_4^s$

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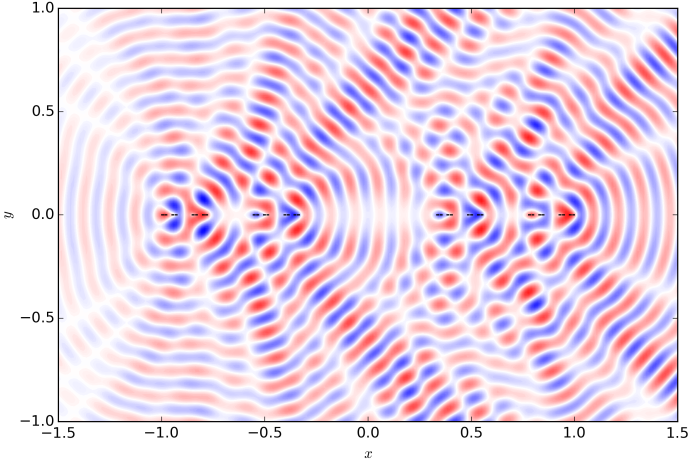

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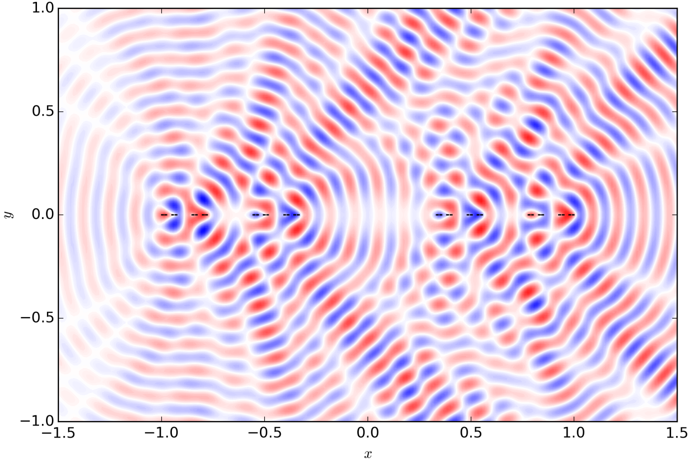
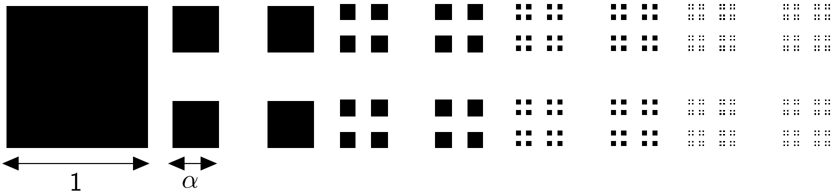


Γ_6 and $\mathrm{Re}\, u_6^s$

Back to Example 3, Cantor dust...

Let $C_\alpha^2 := C_\alpha \times C_\alpha \subset \mathbb{R}^2$ denote the "Cantor dust" $(0 < \alpha < 1/2)$:



C_α^2 is uncountable and closed, with zero area (zero Lebesgue measure).

Furthermore, $\dim_{\mathrm{H}}(C_\alpha^2) = \dfrac{\log(4)}{\log(1/\alpha)} > n - 1 \Leftrightarrow \alpha > 1/4$

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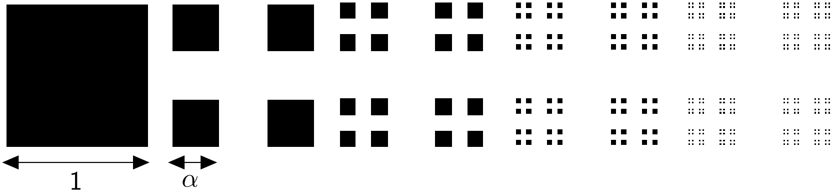

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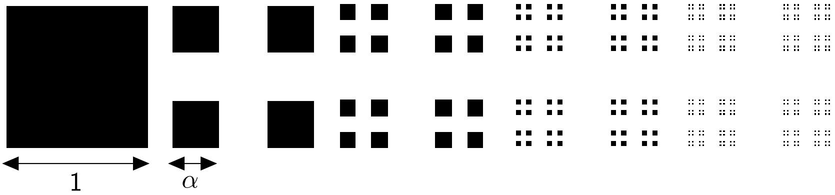

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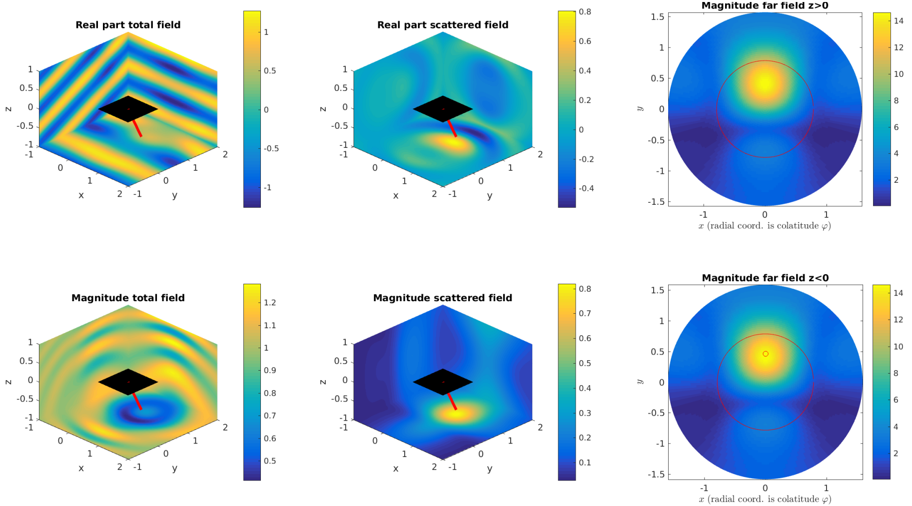
Question: Is the scattered field **zero** or **non-zero** for the sound-soft scattering problem with $\Gamma = C_\alpha^2$?

Answer: **ZERO**, if $0 < \alpha \leq 1/4$; **NON-ZERO**, in general, if $1/4 < \alpha < 1/2$.

Numerical results - Cantor dust $\alpha = 1/3$ ($u^s \neq 0$)

$k = 8$, prefractal level 0



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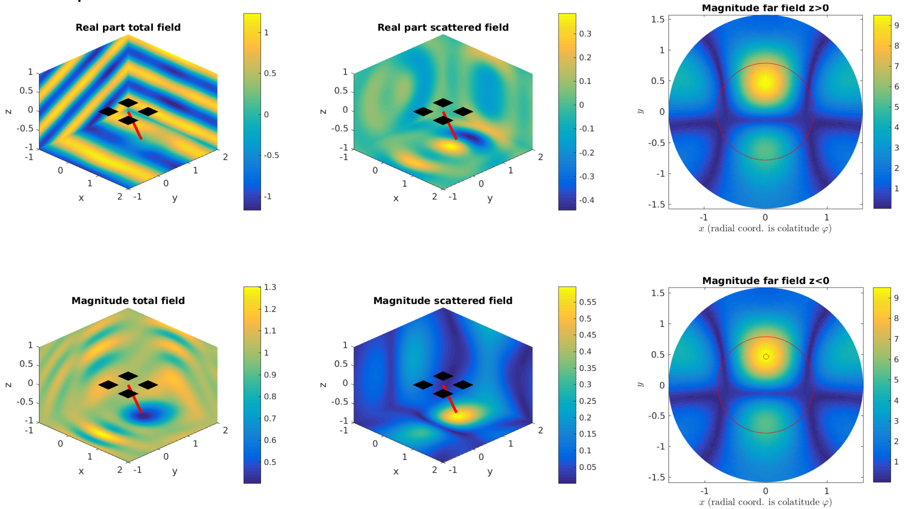
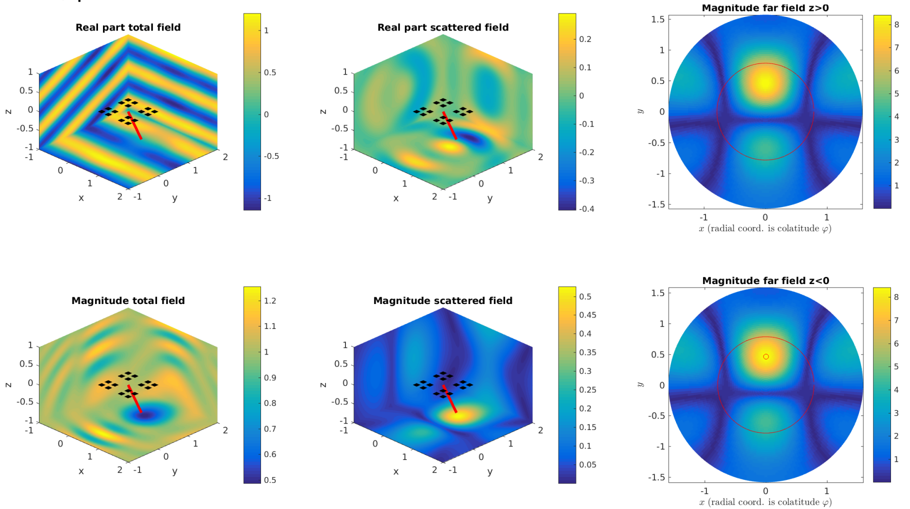


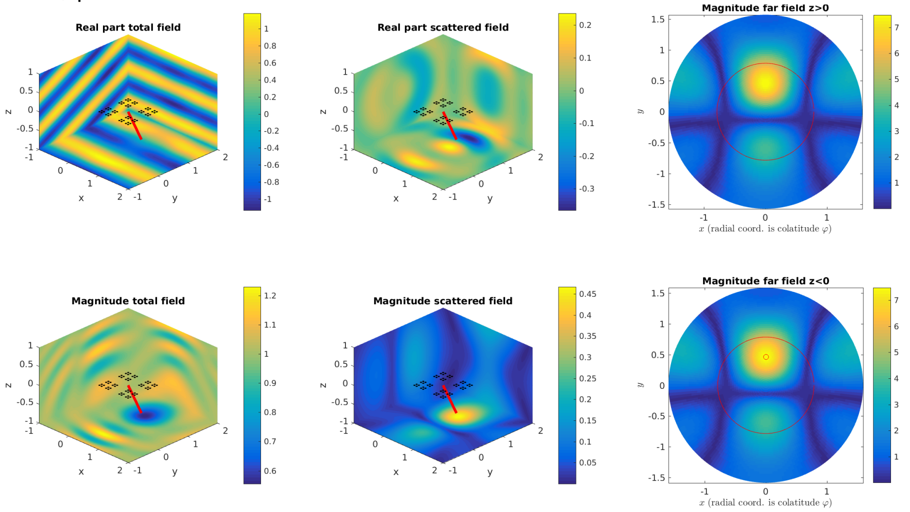
Numerical results - Cantor dust $\alpha = 1/3$ ($u^s \neq 0$)

$k = 8$, prefractal level 2



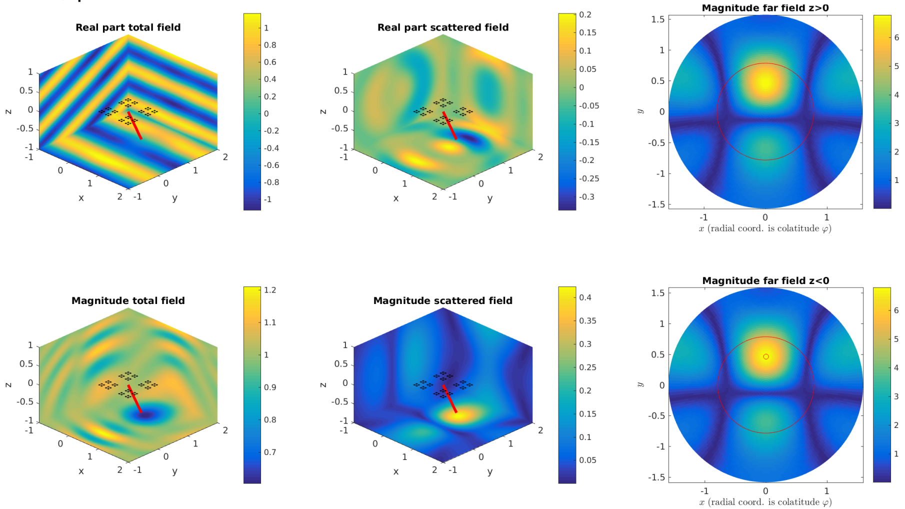
Numerical results - Cantor dust $\alpha = 1/3$ ($u^s \neq 0$)

$k = 8$, prefractal level 3



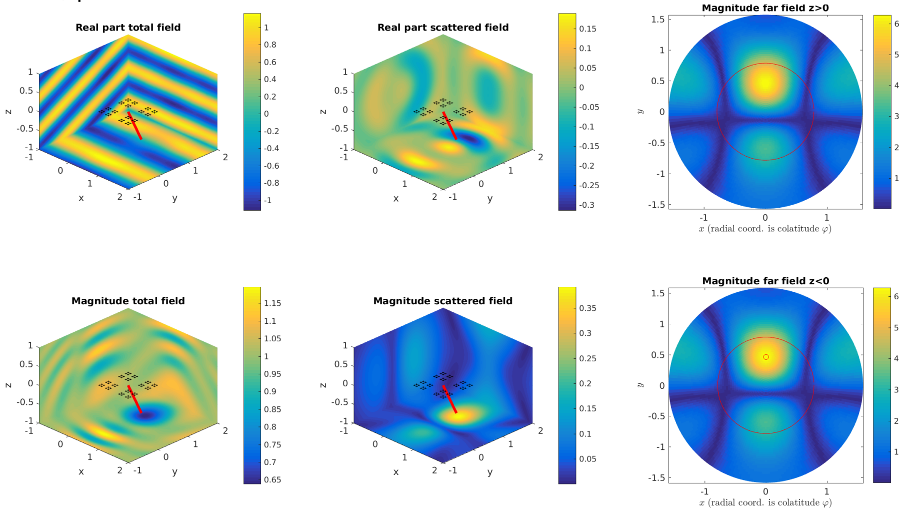
Numerical results - Cantor dust $\alpha = 1/3$ ($u^s \neq 0$)

$k = 8$, prefractal level 4



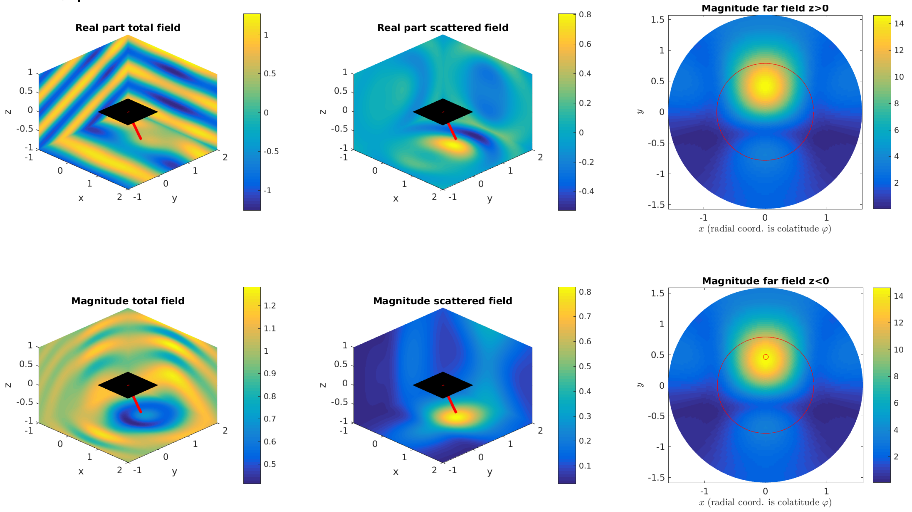
Numerical results - Cantor dust $\alpha = 1/3$ ($u^s \neq 0$)

$k = 8$, prefractal level 5



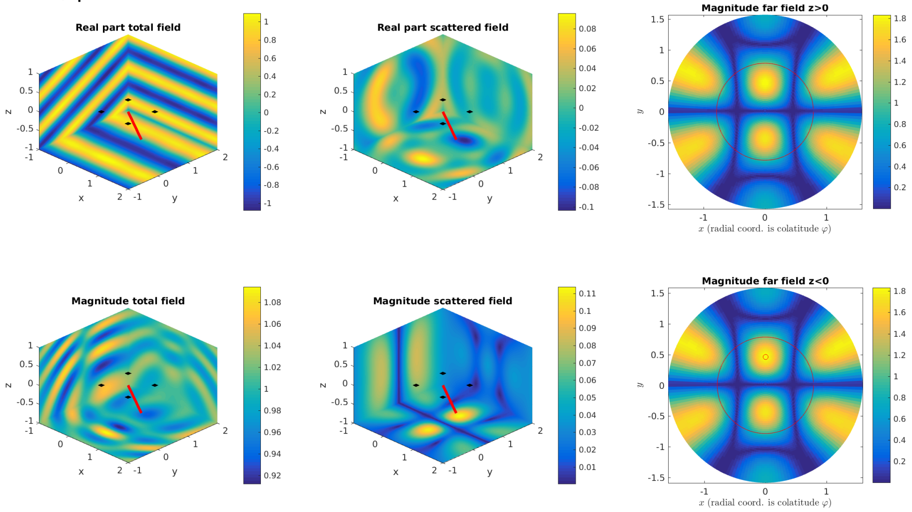
Numerical results - Cantor dust $\alpha = 1/10$ $(u^s = 0)$

$k = 8$, prefractal level 0



Numerical results - Cantor dust $\alpha = 1/10$ $(u^s = 0)$

$k = 8$, prefractal level 1



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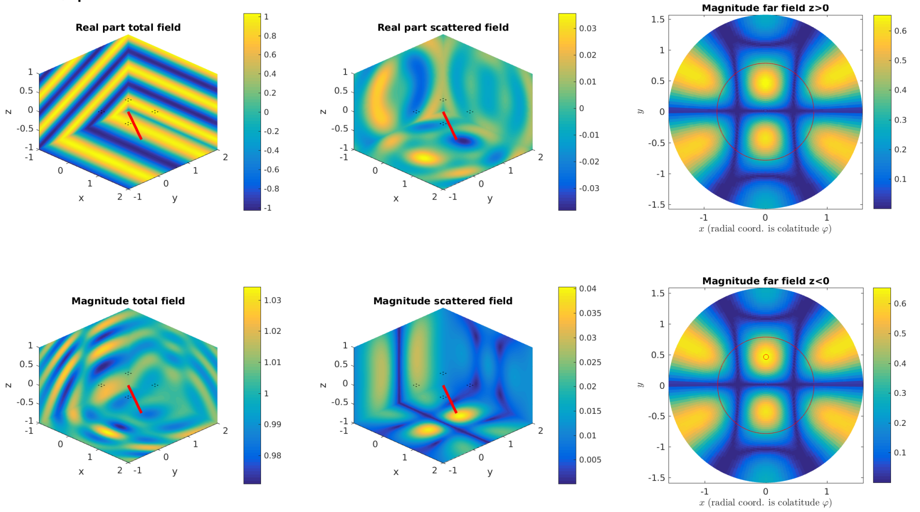

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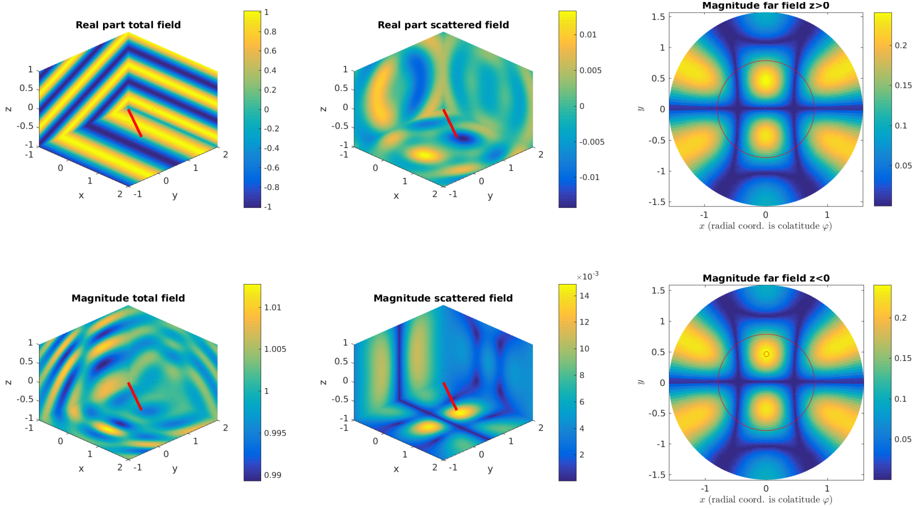

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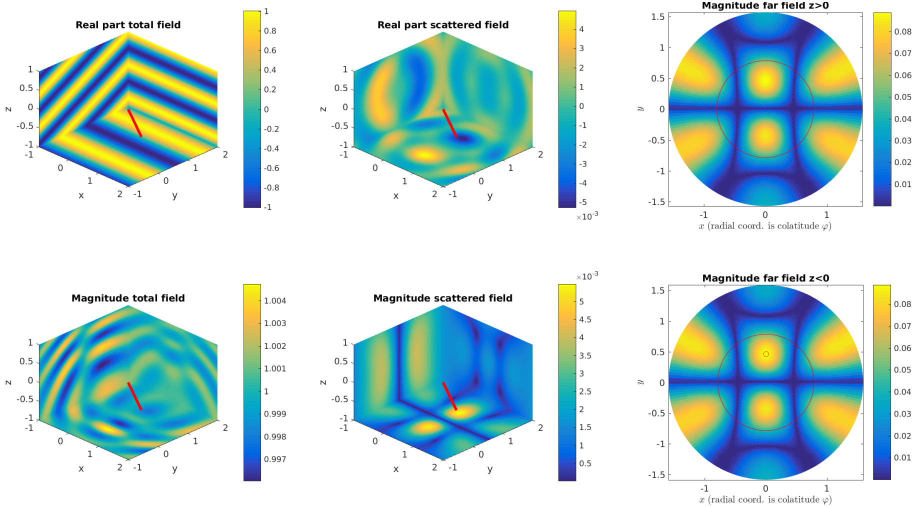

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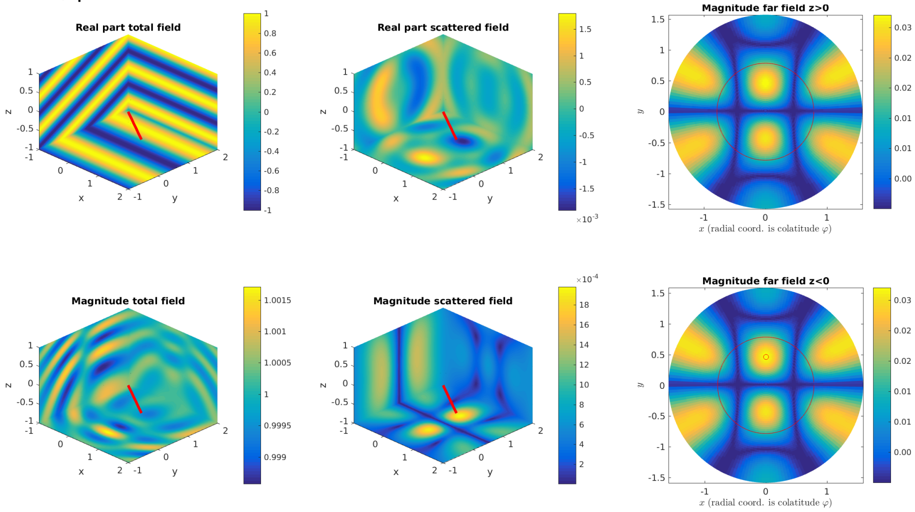

Prefractal convergence from below

Suppose $\Gamma_1 \subset \Gamma_2 \subset \ldots$ are bounded open sets, each sufficiently regular so that solutions of all formulations coincide, e.g

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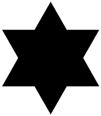
$\Gamma_2 =$ 

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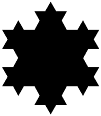
$\Gamma_3 =$ 

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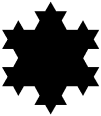
$\Gamma_3 =$ 

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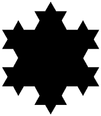
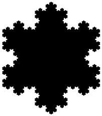
$\Gamma_3 =$  $\Gamma =$  , the **Koch snowflake**.

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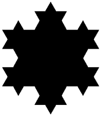
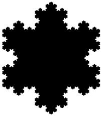
$\Gamma_3 =$  $\Gamma =$ , the **Koch snowflake**.

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$u_j^s \to u^s$ *as* $j \to \infty$, *uniformly on closed subsets of* D, *where* u^s *is the solution to* **BIE-V** *with* $V = \tilde{H}^{-1/2}(\Gamma^\circ)$.

BIE-V: Set $u^s = \mathcal{S}_L\psi$ where $\psi \in V$ satisfies $\langle S_L\psi, \phi\rangle = -\langle u^i|_\Gamma, \phi\rangle$, for $\phi \in V$.

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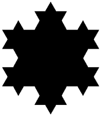
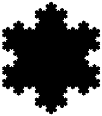
$\Gamma_3 =$  $\Gamma =$ , the **Koch snowflake**.

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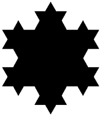
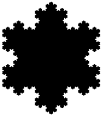
$\Gamma_3 =$  $\Gamma =$  , the **Koch snowflake**.

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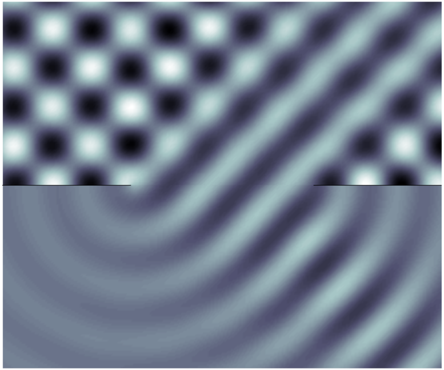
Open problem: Is $\tilde{H}^{-1/2}(\Gamma^\circ) = H_{\overline{\Gamma}}^{-1/2}$ for the **Koch snowflake**?

Variant on Example 1: Infinite sound hard screen with aperture Γ = "Middle Third" Cantor set

By **Babinet's principle**

transmitted field for aperture $\Gamma_j = -$ scattered field for sound soft screen Γ_j.



Aperture Γ_1 and $\mathrm{Re}\, u_1$

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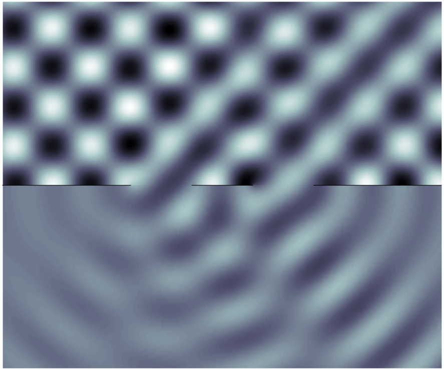

Aperture Γ_2 and $\operatorname{Re} u_2$

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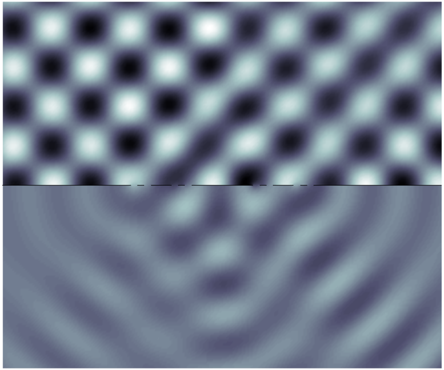

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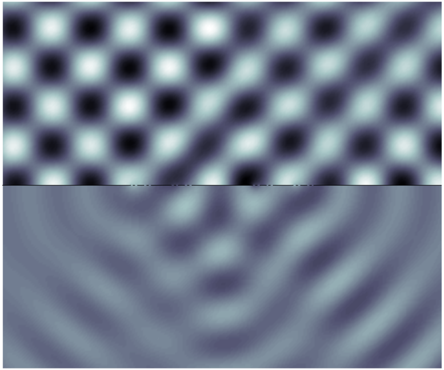

Aperture Γ_4 and $\mathrm{Re}\, u_4$

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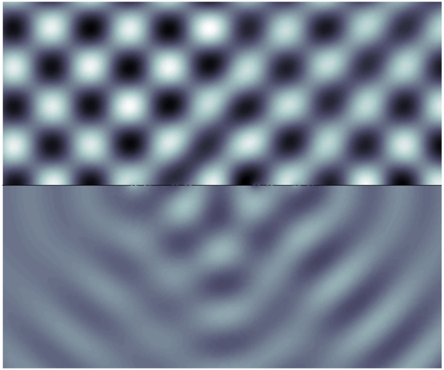

Aperture Γ_5 and $\mathrm{Re}\, u_5$

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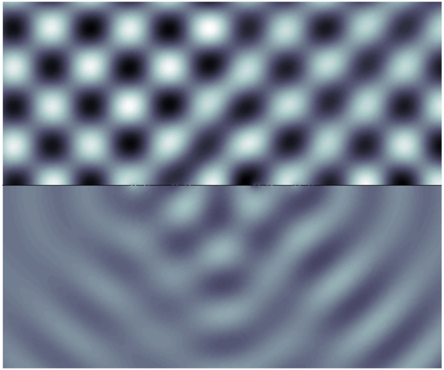

Aperture Γ_6 and $\mathrm{Re}\, u_6$

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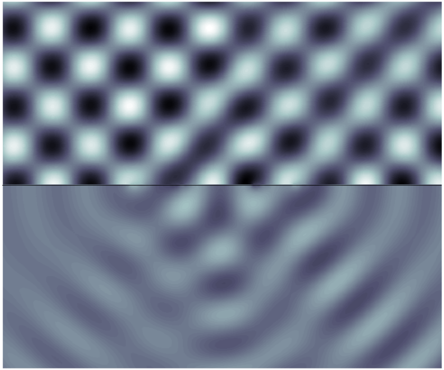

Aperture Γ_7 and $\mathrm{Re}\, u_7$

Recap – a few take homes

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- Sound soft screens with zero surface area can scatter – and then $[\partial_n u] \in H^{-1/2}_{\overline{\Gamma}}$ is **not a function**
- This is interesting and surprising stuff, where subtle properties of Sobolev spaces have physical implications!

References

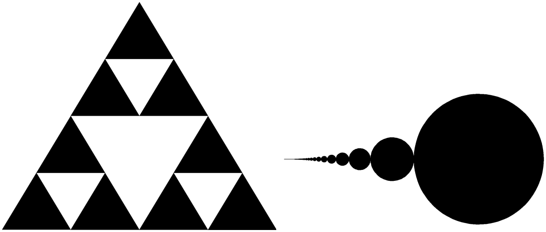
- S. N. Chandler-Wilde & D. P. Hewett, **Well-posed PDE and integral equation formulations for scattering by fractal screens**, 2016, arxiv.org/abs/1611.09539
- S. N. Chandler-Wilde, D. P. Hewett & A. Moiola, **Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to integral equations on fractal screens**, Integr. Equat. Operat. Th., **87**, 179–224, 2017
- D. P. Hewett & A. Moiola, **On the maximal Sobolev regularity of functions supported by subsets of $\mathbb{R}^n$**, Anal. Appl., published online 2016
- S. N. Chandler-Wilde, D. P. Hewett, **Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens**, Integr. Equat. Operat. Th., **82**, 423–449, 2015
- D. P. Hewett, S. Langdon & S. N. Chandler-Wilde & **A frequency independent boundary element method for scattering by two-dimensional screens and apertures**, IMA J. Numer. Anal., **35**, 1698–1728, 2015.

Links and preprints available at `www.reading.ac.uk/∼sms03snc`

What else have we done?

I haven't talked today about:

- Hypersingular integral equations for sound hard fractal screens
- Proving that $\tilde{H}^{\pm 1/2}(\Gamma^\circ) = H^{\pm 1/2}_{\overline{\Gamma}}$ for non-C^0 screens, e.g.

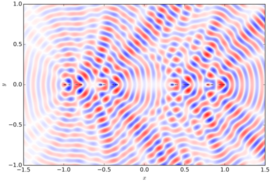


- BVP Formulations that are equivalent to **BIE-V** for each choice of V
- Interpreting **BIE-V** as an equation $S\psi = -P_{V^*}u^i$, where $S : V \to V^*$
- "Swiss Cheese" screens!

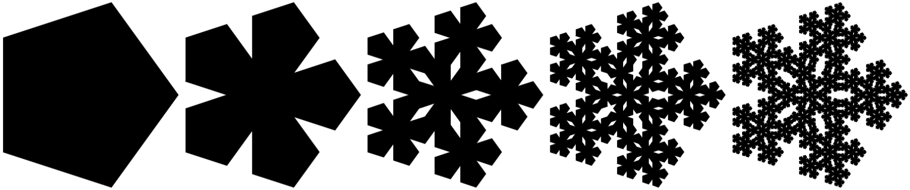
See the references, or talk to me or Dave, for details.

Many Open questions

- At what rate do prefractal solutions converge?
- Numerical analysis in the joint limit of prefractal level and mesh refinement?
- Regularity results for fractal solution?
- Curved screens?
- Maxwell case?



- Inverse problems? ...
- . . .



A Final Reference

Lord Rayleigh, “Theory of Sound”, 2nd Ed., Vol. II, Macmillan, New York, 1896: the 19th century mathematics of screens and apertures

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If $P\cos(nt+\epsilon)$ denote the value of $d\phi/dx$ at the various points of the area (S) of the aperture, the condition for determining P and ϵ is by (6) § 278,

$$-\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS=\cos nt \quad\text{.........(2)},$$

where r denotes the distance between the element dS and any fixed point in the aperture. When P and ϵ are known, the complete value of ϕ for any point on the positive side of the screen is given by

$$\phi=-\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS \quad\text{..........................(3)},$$

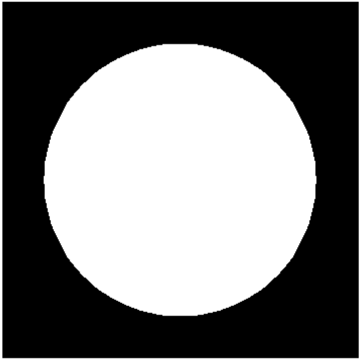
and for any point on the negative side by

$$\phi=+\frac{1}{2\pi}\iint P\,\frac{\cos(nt-kr+\epsilon)}{r}\,dS+2\cos nt\cos kx \quad\text{......(4)}.$$

The expression of P and ϵ for a finite aperture, even if of circular form, is probably beyond the power of known methods; but in the

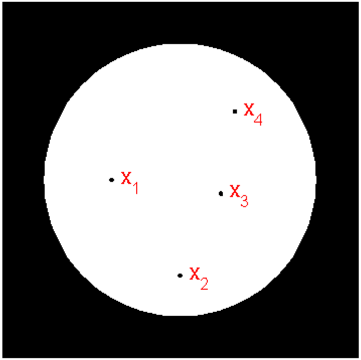
This is precisely **BIE-V**, admittedly not worrying about fractals or function spaces!

Example 4: “Swiss cheese” aperture in sound soft screen



Infinite sound soft $(u = 0)$ screen with circular aperture of radius one:

Example 4: "Swiss cheese" aperture in sound soft screen



Infinite sound soft $(u = 0)$ screen with circular aperture of radius one:

- Take a sequence of points x_1, x_2, x_3, ... that are **dense** in the aperture

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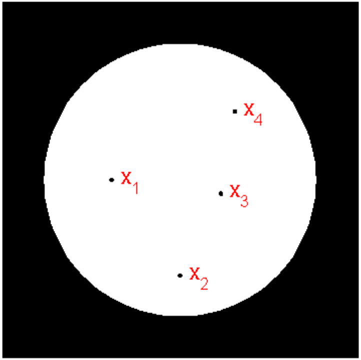

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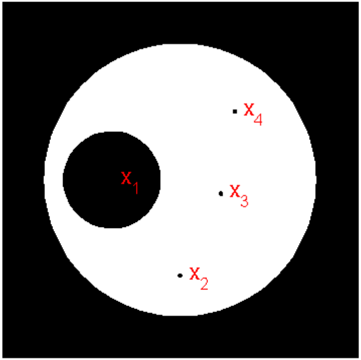

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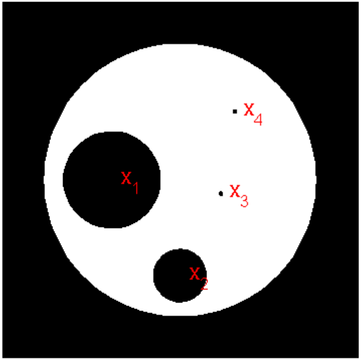

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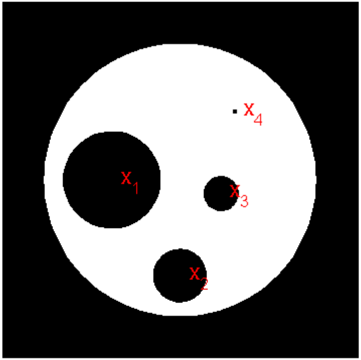

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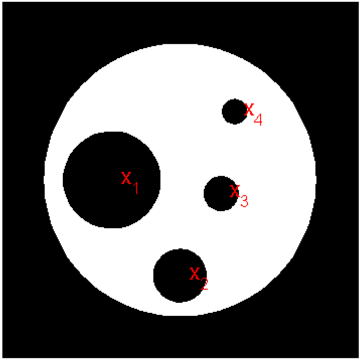

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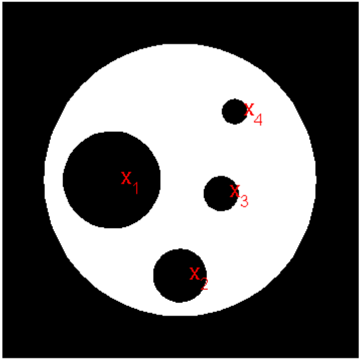

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Question: Is the transmitted field **zero** or **non-zero** in the limit? (The limiting aperture is a so-called **Swiss cheese**.)

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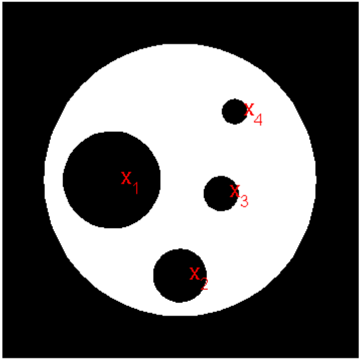

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Argument A: Limiting Swiss cheese aperture has area $A \geq \pi(1 - r_1^2 - r_2^2 ...)$. If $A > 0$ then sound transmitted?

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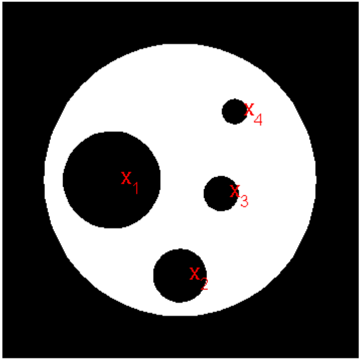

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Argument B: Limiting aperture has **empty interior** and u is continuous so $u = 0$ also on the aperture and so no transmitted wave?