Integral Equations for Wave Scattering: Numerical Solution and Wavenumber-Explicit Bounds

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Joint work with collaborators, notably:
Ivan Graham, Euan Spence (Bath), Steve Langdon (Reading)

Recent Advances in the Numerical Analysis of PDEs:
Celebrating the 65th birthday of Prof Ivan Graham

Overview

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by integral equation methods.

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And what I've worked on for much of my career, partly collaborating with Ivan

(ii) understanding how everything depends on k.

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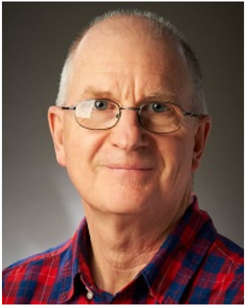

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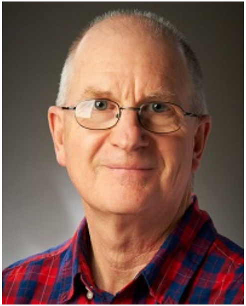

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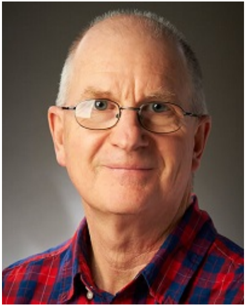

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I'll talk today about two specific acoustics problems.

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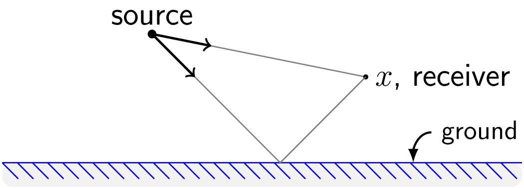
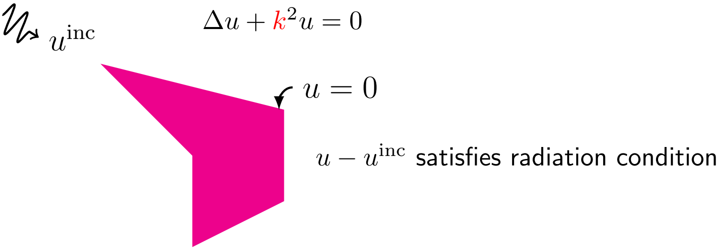


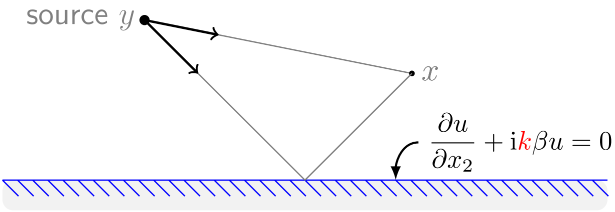
Problem 2



Problem 1: Sound Propagation over Inhomogeneous Impedance Plane

Problem I've worked on since 1985, 11 papers in *J Sound Vib*, *IMA J Numer Anal*, *Math Meth Appl Sci*, *J Math Anal Appl*, *Numer Math*, *Proc R Soc A*, *SINUM*, the last two with **Steve Langdon** (Ivan's 1999 PhD student).

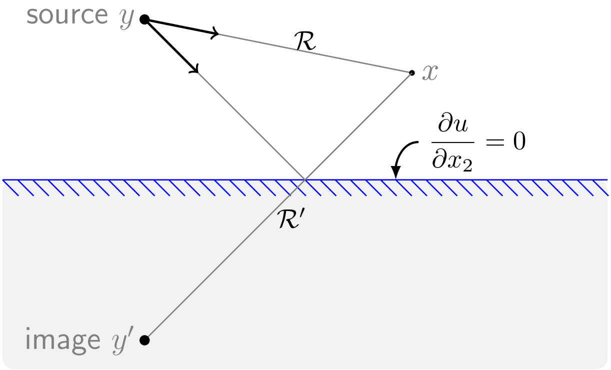
$$\Delta u + k^2 u = -\delta_y$$



- $k > 0$ is the **wavenumber**
- $\beta \in L^\infty(\mathbb{R})$, the **impedance**, is typically piecewise constant, and $\Re\beta \geq 0$
- u satisfies standard radiation condition at infinity

Problem 1: Simplest Case, $\beta \equiv 0$

$$\Delta u + k^2 u = -\delta_y$$



Solution by method of images is

$$u(x) = G_0(x, y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R}) + \frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R}')$$

where $H_0^{(1)}$ is a Hankel function.

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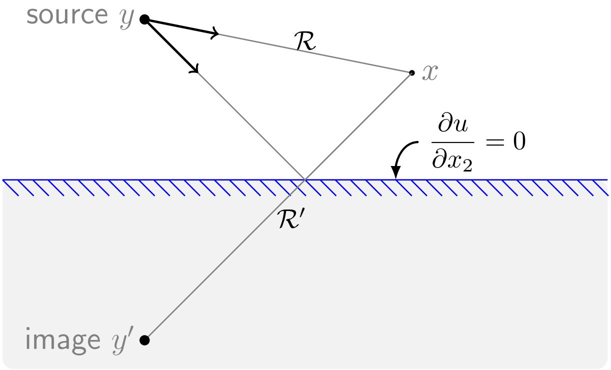

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N.B. $G_0(x,y)$ is just the **Neumann Green's function**.

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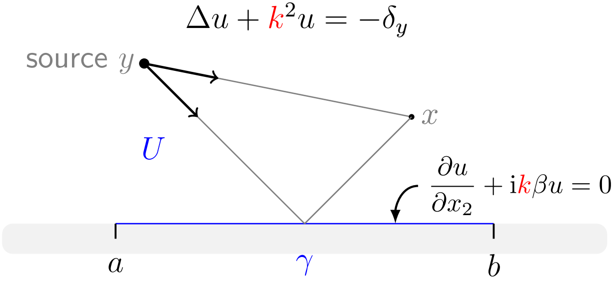

By Green's theorem – $G_0(x, y)$ is the Neumann Green's function –

$$u(x) = G_0(x, y) + \int_\gamma G_0(x, z)\mathrm{i}k\beta(z)u(z)\mathrm{d}s(z), \quad x \in \overline{U}.$$

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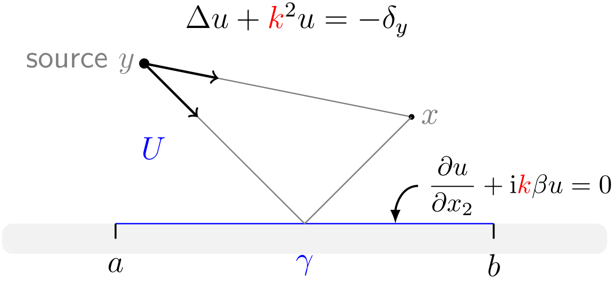

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In particular, where $\phi(x_1) := u((x_1, 0))$, and if $y = (0, y_2)$,

$$\phi(x_1) = \frac{\mathrm{i}}{2}H_0^{(1)}\left(k\sqrt{x_1^2 + y_2^2}\right) - \frac{k}{2}\int_a^b H_0^{(1)}(k|x_1 - z_1|)\beta(z_1)\phi(z_1)\mathrm{d}z_1, \quad a \leq x_1 \leq b.$$

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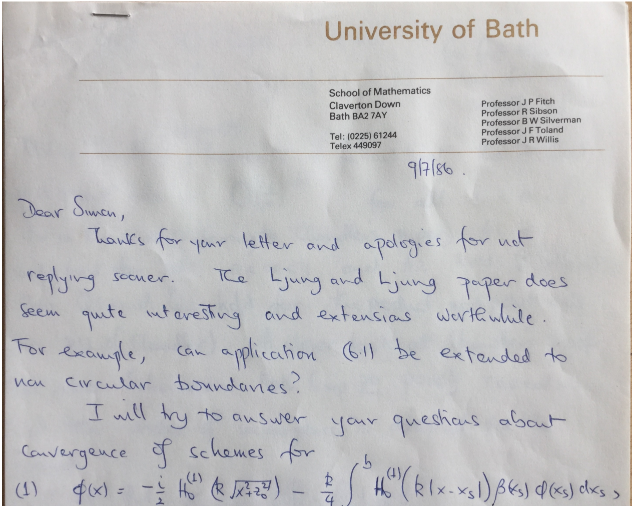

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Professor J F Toland
Professor J R Willis

9/7/86.

Dear Simon,

Thanks for your letter and apologies for not replying sooner. The Ljung and Ljung paper does seem quite interesting and extensions worthwhile. For example, can application (6.1) be extended to non circular boundaries?

I will try to answer your questions about convergence of schemes for

(1) $\phi(x) = -\frac{i}{2} H_0^{(1)}(k\sqrt{x^2+z_0^2}) - \frac{k}{4}\int^b H_0^{(1)}(k|x-x_s|)\beta(x_s)\phi(x_s)\,dx_s$,

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Ivan perfect person to consult: expert (with **Kendall Atkinson**, **Ian Sloan**, Graeme Chandler, Claus Schneider, ...) on methods for integral equations with kernels like $H_0^{(1)}(k|t-s|) = \log|t-s|A(s,t) + B(s,t)$ where A, B are smooth.

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Also an expert in the singularities in these equation. E.g. Graham (1982), if

$$\phi(t) = f(t) + \int_0^1 \kappa(t-s)\phi(s)\mathrm{d}s, \quad 0 \le t \le 1,$$

and $f \in L_1[0,1]$ and κ is in a Nikol'skii space intermediate between $L_1[-1,1]$ and $W_1^1[-1,1]$, then, for $m \in \mathbb{N}$,

$$\phi = \text{finite linear combination of known singularities} + \psi,$$

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with $\psi \in W_1^m[0,1]$. E.g., if $f \in C^\infty[0,1]$ and $\kappa(s-t) = H_0^{(1)}(c|s-t|)$ then

$$\phi(s) = \phi(0)s(x) + \phi(1)s(1-x) + \psi(s),$$

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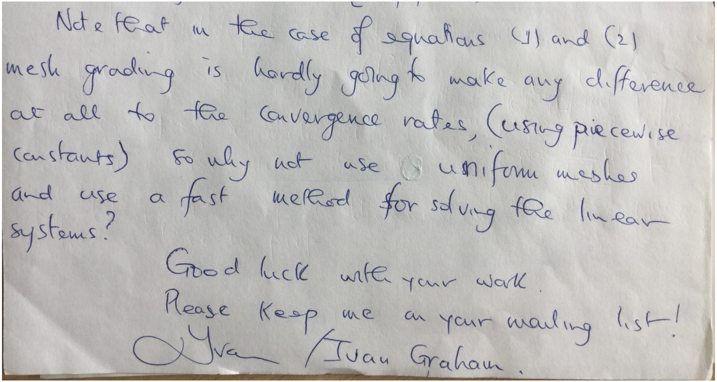

Note that in the case of equations (1) and (2) mesh grading is hardly going to make any difference at all to the convergence rates, (using piecewise constants) so why not use uniform meshes and use a fast method for solving the linear systems?

Good luck with your work.

Please keep me on your mailing list!

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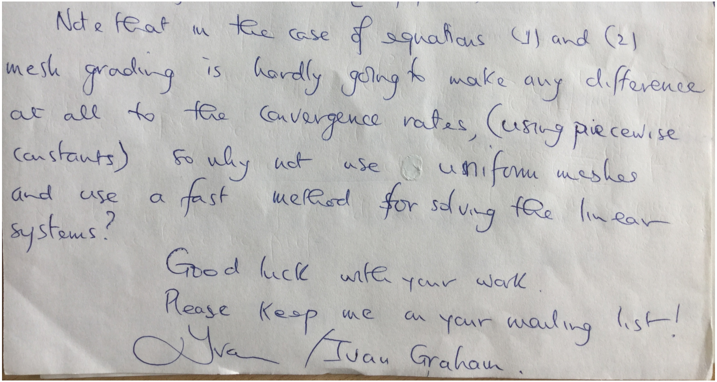

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Numerical method ingredients:

- Uniform grid, stepsize $h = (b - a)/N$
- Approximate $\beta \in L^\infty[a, b]$ by its local average on each grid subinterval
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Theorem (C-W, Rahman, Ross 2002: plane wave incidence)

Suppose β takes values in compact subset Q of the right-hand complex plane. Then, after 7 iterations, provided $kh \le c_Q$,

$$\|\phi - \phi_h\|_\infty \le C_Q kh \log(1/kh),$$

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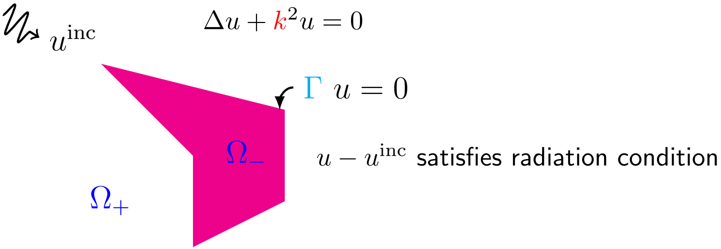
This my first k-explicit estimate, and an early k-explicit estimate for BEM for wave scattering (cf. Löhndorf, Melenk 2011, Graham, Löhndorf, Melenk, Spence 2015).

Problem 2: Scattering by Sound Soft Obstacles: Integral Equations and k-Explicit Bounds on the Operators



Assume throughout that Ω_- is bounded and Lipschitz.

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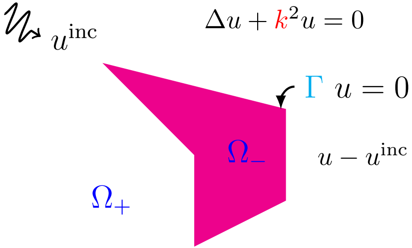

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$u - u^{\mathrm{inc}}$ satisfies radiation condition

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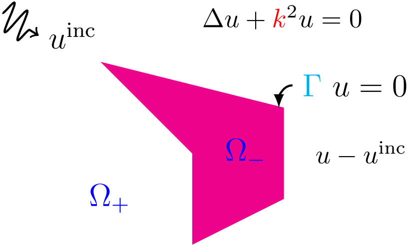
Theorem (Green's Representation Theorem)

$$u(x) = u^{\mathrm{inc}}(x) + \int_\Gamma \Phi(x,y)\partial_n^+ u(y)\, ds(y), \quad x \in \Omega_+.$$

where

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ (2D)}, \quad := \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$



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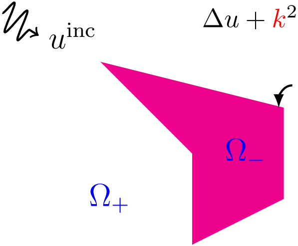

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Taking a linear combination of Dirichlet (γ_+) and Neumann (∂_n^+) traces, we get the **boundary integral equation** (Burton & Miller 1971)

$$\frac{1}{2}\partial_n^+ u(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \partial_n^+ u(y) ds(y) = f(x), \quad x \in \Gamma,$$

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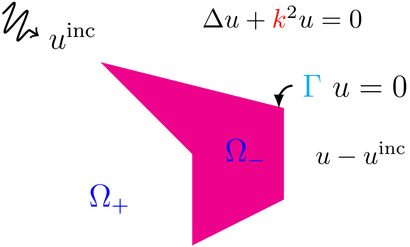
$$f := \partial_n^+ u^{\text{inc}} + \mathrm{i}\eta\gamma_+ u^{\text{inc}}.$$



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in operator form $A_{k,\eta}\partial_n^+ u = f := \partial_n^+ u^{\text{inc}} + \mathrm{i}\eta\gamma_+ u^{\text{inc}}$.

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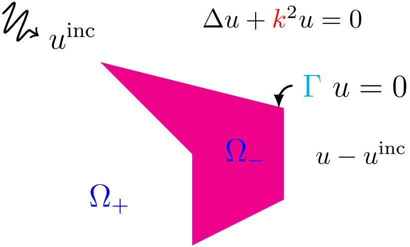

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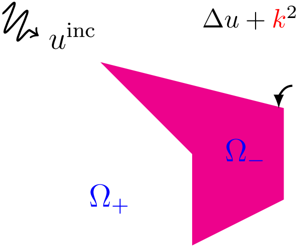
If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$.



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If $\eta \approx k$ and Ω_- is: (i) star-shaped with respect to a ball and piecewise smooth; or (ii) C^∞ and non-trapping; then, as an operator on $L^2(\Gamma)$, for $k \geq k_0$,

$$\|A_{k,\eta}^{-1}\| \lesssim 1, \quad \|A_{k,\eta}\| \lesssim k^{1/2}\log k, \quad \mathrm{cond}\, A_{k,\eta} \lesssim k^{1/2}\log k.$$

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This k-dependence translates to BEM discretisations.

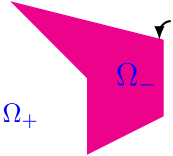
k	$\|A_{k,k}\|$	p	$\|A_{k,k}^{-1}\|$
5	3.089		1.024
10	3.611	0.23	1.024
20	4.608	0.35	1.023
40	6.032	0.39	1.023
80	8.117	0.43	1.023
160	11.068	0.45	1.023
320	15.253	0.46	1.023
640	21.177	0.47	1.023

Galerkin BEM discretisations for a square of sidelength 2, using piecewise constants, 10 elements per wavelength, from Betcke, C-W, Graham, Langdon, Lindner (2011)

How do you bound $\|A_{k,k}^{-1}\|$ for $k \geq k_0$? (Recipe from Baskin, Spence, Wunsch 2016)

$\Delta u + k^2 u = f \in L^2(\Omega_+)$, compactly supported

Γ $u = g \in H^1(\Gamma)$



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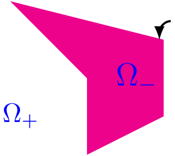

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Step 1 (Resolvent Estimate). Show that, for every $R > 0$, if $g = 0$,

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_+)},$$

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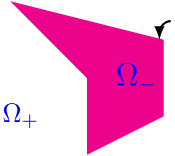

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Step 2 (DtN Map Bound). It follows that, if $f = 0$,

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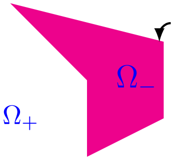

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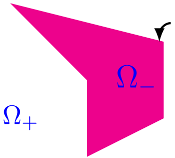

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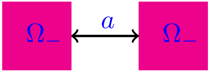
Sharper by a factor $k^{1/2}$ if Ω_- is starlike (Melenk 1995), or is C^∞ (Baskin et al 2016)

First bound on $A_{k,k}^{-1}$ for Ω_- trapping (using this recipe).

This bound is for *neutrally trapping* obstacles, e.g.



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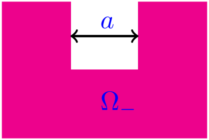

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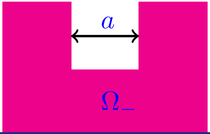

Theorem (C-W, Graham, Langdon, Lindner 2009, C-W, Spence 2017)

For trapping domains like this (neutrally trapping) it holds, for $k \geq k_0$, that

$$\|A_{k,k}^{-1}\| \lesssim k^{5/2}, \text{ and that } k^{9/10} \lesssim \|A_{k,k}^{-1}\|$$

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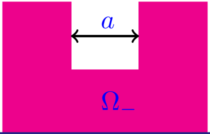

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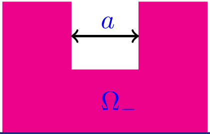

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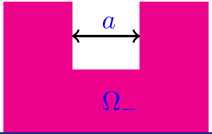

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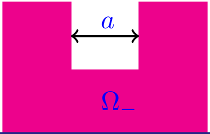

Theorem (C-W, Graham, Langdon, Lindner 2009, C-W, Spence 2017)

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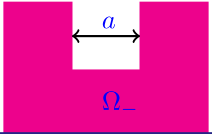

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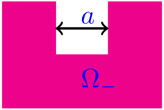
- The recipe can also be used (with Burq 1998) to get worst case bound
- Numerical results suggest the lower bound is sharper



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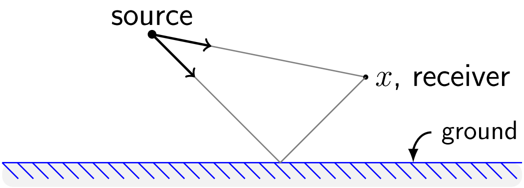
k	$\|A_{k,k}\|$	p	$\|A_{k,k}^{-1}\|$	p
5	4.835		1.969	
10	5.201	0.11	3.121	0.66
20	5.629	0.11	5.539	0.83
40	6.182	0.14	10.322	0.90
80	8.112	0.39	19.774	0.94
160	11.066	0.45	38.351	0.96
320	15.254	0.46	75.156	0.97

Galerkin BEM discretisations for the trapping domain with ka a multiple of π, using piecewise constants, 10 elements per wavelength, from Betcke, C-W, Graham, Langdon, Lindner (2011)

Conclusions

I've said something about integral equation formulations and k-explicit estimates for two (representative) problems in acoustics that have connected me to Ivan over the last 31 years.

Problem 1



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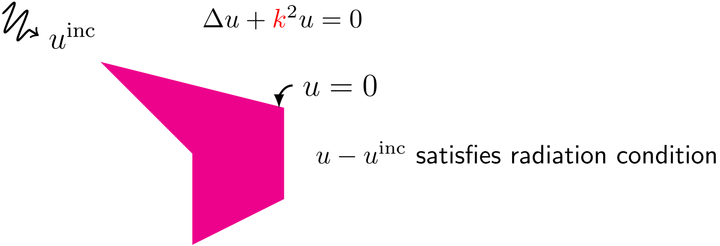
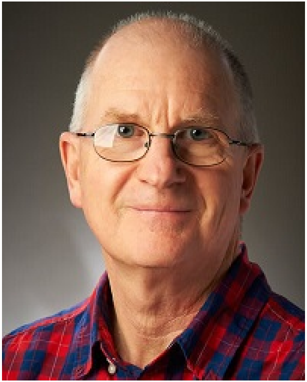


Conclusions

Ivan thank you for your friendship over 31 years ... and Happy 65th Birthday!



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