Wave scattering by trapping obstacles: resolvent estimates and applications to boundary integral equations and their numerical solution

Simon Chandler-Wilde

Department of Mathematics and Statistics
University of Reading
s.n.chandler-wilde@reading.ac.uk

Joint work with:
Euan Spence (Bath), Andrew Gibbs (Reading/Leuven), Valery Smyshlyaev (UCL)

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More info: new preprint “High-frequency bounds ...” on arXiv

The wave equation and Helmholtz equation

In acoustics the increase in air pressure at x at time t, $U(x,t)$, satisfies

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right).$$

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where $u(x) = A(x)\exp(\mathrm{i}\phi(x))$ satisfies the **Helmholtz equation**

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with $k = \omega/c$ the **wavenumber**. E.g. if $u(x) = \exp(\mathrm{i}kx\cdot d)$, for some **unit vector** d, then

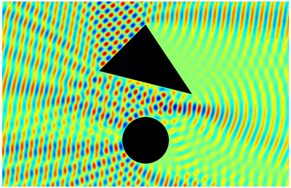
$$U(x,t) = \Re\left(u(x)\mathrm{e}^{-\mathrm{i}\omega t}\right) = \cos(kx\cdot d - \omega t)$$

is a **plane wave** travelling in direction d with **wavelength**

$$\lambda = 2\pi/k = c/f.$$

Challenges of $\Delta u + k^2 u = 0$ when k is large

1. Solution is oscillatory and multiscale: one scale is the wavelength $\lambda = 2\pi/k$.



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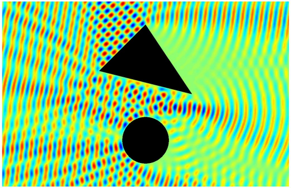

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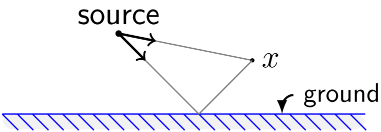

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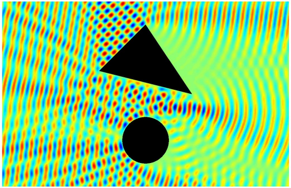

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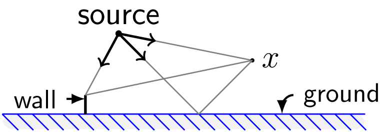

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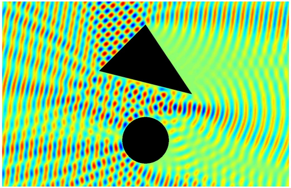

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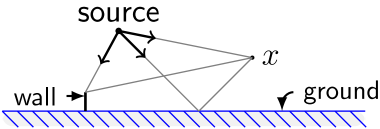
where sum over **rays** passing through x, with



$\arg u_j(x)$ = **optical length** of ray path $= ks_j$

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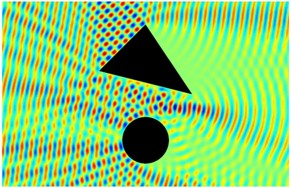

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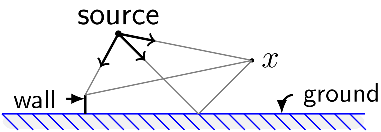

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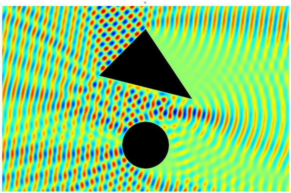

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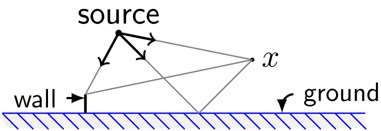

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The rigorous justification of such approximations is the concern of **semi-classical** analysis.

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection** and **diffraction** events.

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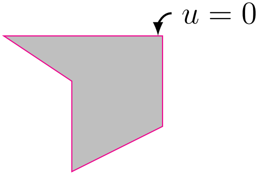
u satisfies Sommerfeld rad. cond. (SRC)

$\Delta u + k^2 u = f$ (source, compactly supported)



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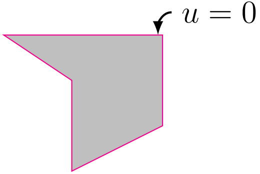

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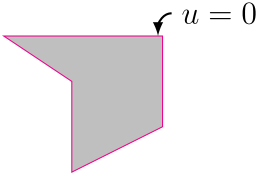

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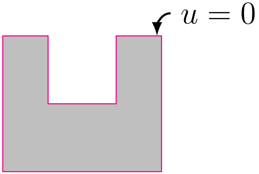

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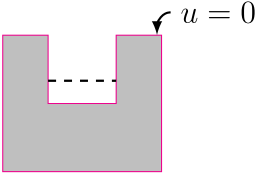

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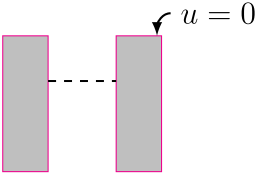

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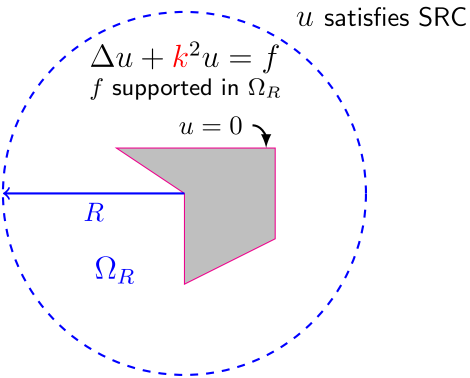
But particularly about cases where the obstacle is **trapping** supporting a **trapped ray/billiard trajectory**.

Including cases where the obstacle has more than one component, in other words **multiple scattering**.

Overview of Talk

1. Helmholtz equation: what is it and why interesting
2. What is this talk about?
3. Resolvent estimates
   - What are they?
   - The three known estimates and their geometries
   - A new estimate for parabolic trapping
   - The Morawetz/Rellich identity method of proof
4. Implications for Boundary Integral Equations
5. Implications for hp-BEM
6. Conclusions

What is a resolvent estimate?



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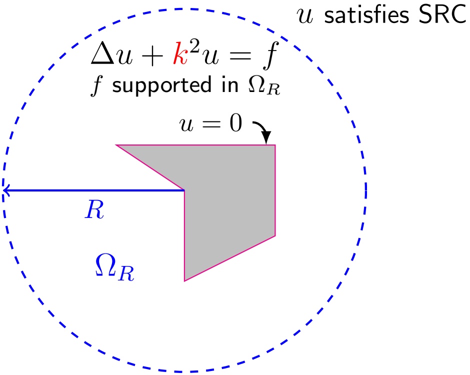

It is the wavenumber-explicit bound that, for $R > 0$, and some specified $c(k)$,

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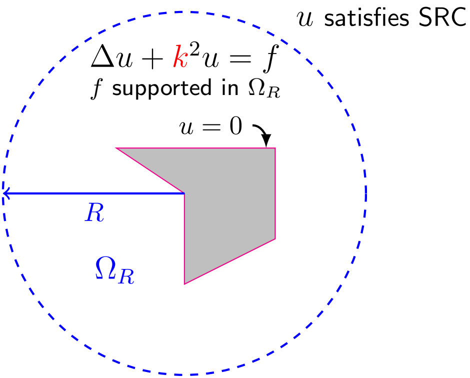

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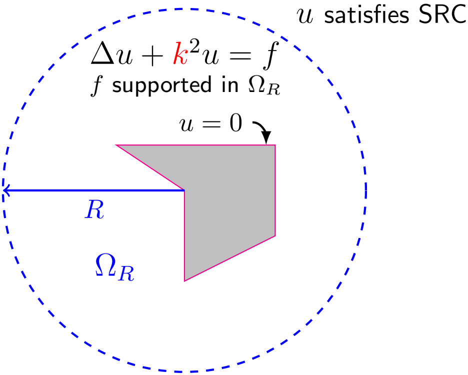

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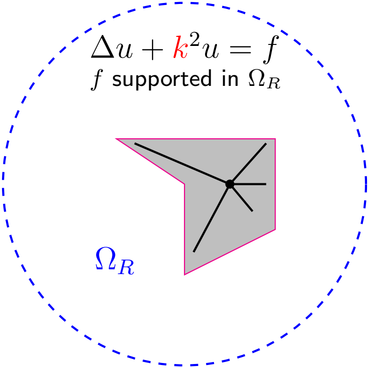
We will see that resolvent estimates give us: bounds on **DtN maps**, on inverses of **boundary integral operators**, on errors in **FEM**, **BEM**, ...

The known estimates and their geometries



Star-shaped obstacle (C^∞: Morawetz 1975; C^0: C-W & Monk 2008)

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim \|f\|_{L^2(\Omega_R)}, \quad \text{i.e. } c(k) = 1$$

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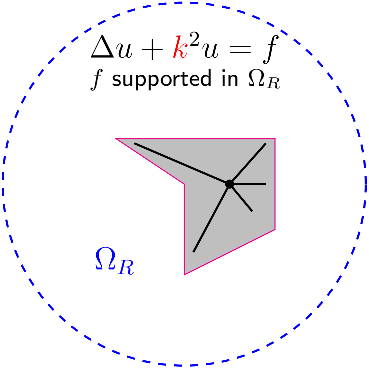

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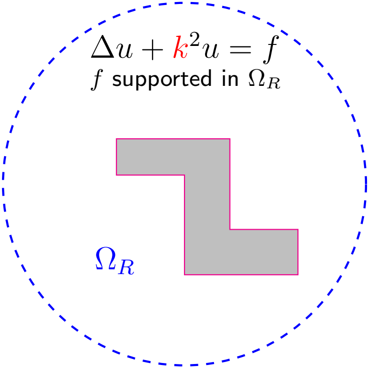

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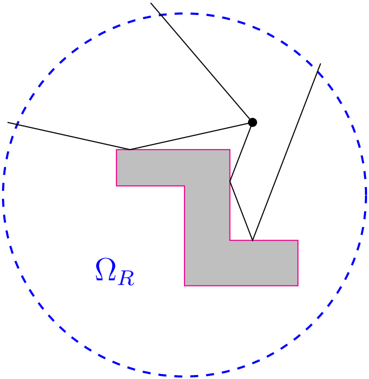

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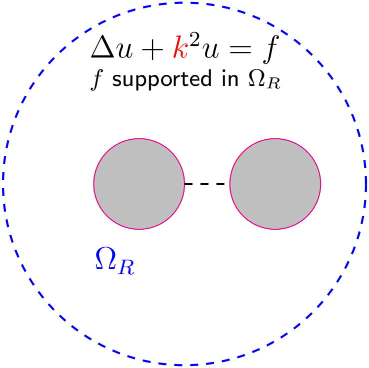
Nontrapping: there exists $T > 0$ such that all the billiard trajectories starting in Ω_R at time zero and travelling at unit speed leave Ω_R by time T.

The known estimates and their geometries



Two or more C^∞ strictly convex, positive curvature obstacles
(Ikawa 1988, Burq 2004), example of **hyperbolic, unstable trapping**

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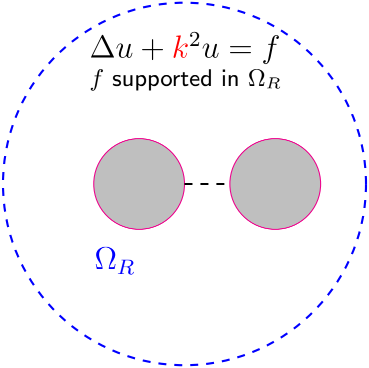

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so **only logarithmically worse** than the nontrapping case.

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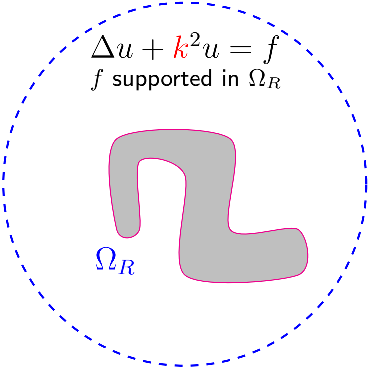

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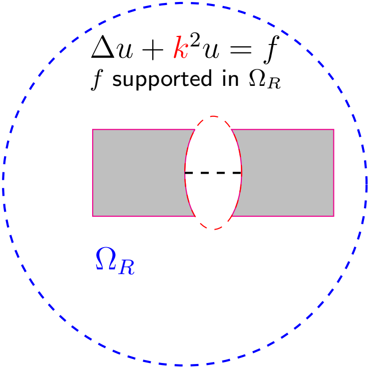

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This achieved for some $k_m \to \infty$ when there is **elliptic, stable trapping** (Cardoso, Popov 2002; Betcke, C-W, Graham, Langdon, Lindner 2011) with a **quasimode localised around the trapped ray**.

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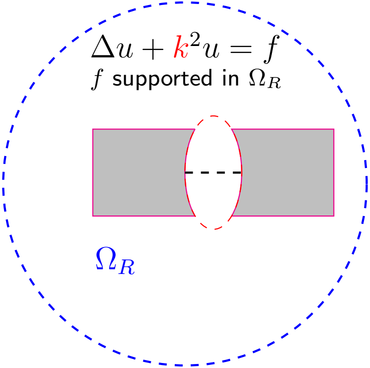

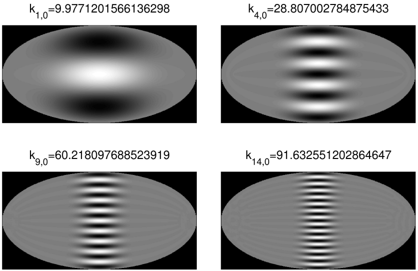

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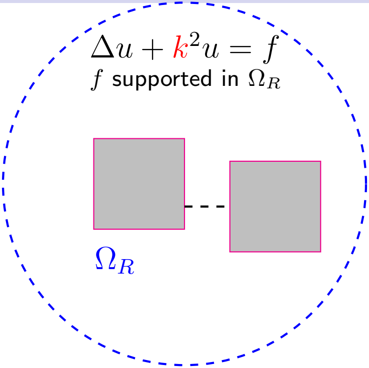
Where have we got to in the talk?

Our **new estimate** for **parabolic, neutral** trapping



Theorem (C-W, Spence, Gibbs, Smyshlyaev 2017)

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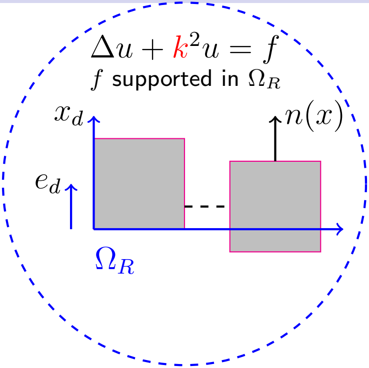

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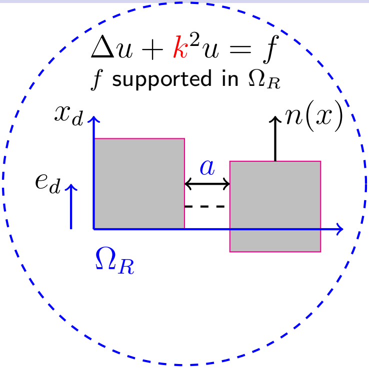
Applies to a general Lipschitz obstacle class, in particular when

$$x_d e_d \cdot n(x) \geq 0 \quad \textit{on the boundary}$$

Our **new estimate** for **parabolic, neutral** trapping



Theorem (C-W, Spence, Gibbs, Smyshlyaev 2017)

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim k^2 \|f\|_{L^2(\Omega_R)}, \quad \textit{i.e. } c(k) = k^2.$$

Applies to a general Lipschitz obstacle class, in particular when

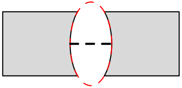
$$x_d e_d \cdot n(x) \geq 0 \quad \textit{on the boundary}$$

Further, $\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \gtrsim k\|f\|_{L^2(\Omega_R)}$, *for* $k = m\pi/a$, $m = 1, 2, \ldots$.

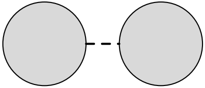
Recap of resolvent estimates for trapping obstacles

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0,$$

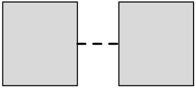
where $c(k) = 1$ for **nontrapping** obstacles, and



$c(k) = \exp(\alpha k)$
elliptic



$c(k) = \log(2 + k)$
hyperbolic



$c(k) = k^2$
parabolic

How are these resolvent estimates proved?

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The Morawetz/Rellich identity method

Used for:

- Star-shaped obstacles (Morawetz 1975, C-W, Monk 2008)
- "Nearly all" nontrapping obstacles in 2-d (Morawetz, Ralston, Strauss 1977)
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Cathleen Morawetz (1923-2017), Courant Institute, New York.

Listen to the interviews at
https://www.simonsfoundation.org/2012/12/20/cathleen-morawetz/
e.g. on women in mathematics, working with Courant, Courant and flexible working, the founding of the Courant Institute, ...

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Define **Morawetz multiplier** $\mathcal{Z}u$ by

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where Z, α, β are real-valued, with $Z \cdot n \geq 0$ on the boundary

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$$\int_{\Omega_R} \left(|\nabla u|^2 + k^2 |u|^2\right) \,\mathrm{d}x = -2\Re \int_{\Omega_R} \overline{\mathcal{Z}u} f \,\mathrm{d}x - \int_{\partial\Omega_R} +\text{ve} \leq \epsilon \|\mathcal{Z}u\|^2_{L^2(\Omega_R)} + \epsilon^{-1} \|f\|^2_{L^2(\Omega_R)}.$$

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In **rough surface scattering** (C-W, Monk 2005) use $Z(x) = x_d e_d$, $\alpha = 1/2$, $\beta = R$, to get

$$\int_{\Omega_R} |\partial_d u|^2 \, \mathrm{d}x \leq -2\Re \int_{\Omega_R} \overline{\mathcal{Z}u} f \, \mathrm{d}x \leq \epsilon \|\mathcal{Z}u\|^2_{L^2(\Omega_R)} + \epsilon^{-1} \|f\|^2_{L^2(\Omega_R)};$$

then use Friedrichs inequality to bound $\|u\|_{L^2(\Omega_R)}$ in terms of $\|\partial_d u\|_{L^2(\Omega_R)}$.

How is our new estimate for parabolic trapping proved?

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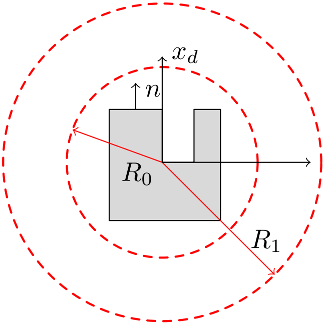

Choose $R_1 > R_0 > 0$ and set

$$Z(x) = x_d e_d \text{ for } |x| \leq R_0, \quad Z(x) = x \text{ for } |x| \geq R_1.$$

Resolvent estimate obtained if $Z \cdot n = x_d e_d \cdot n \geq 0$ on boundary & $R_1/R_0 \geq 121$.

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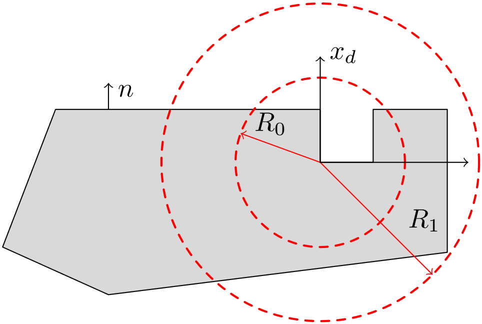

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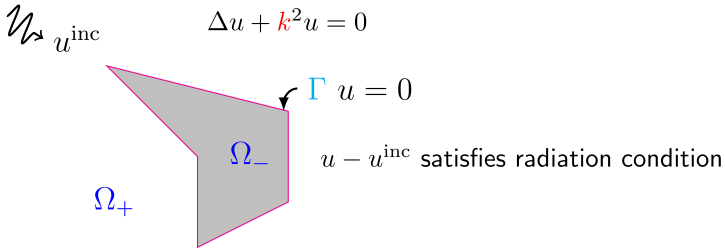
Overview of Talk

1. Helmholtz equation: what is it and why interesting
2. What is this talk about?
3. Resolvent estimates
   - What are they?
   - The three known estimates and their geometries
   - A new estimate for parabolic trapping
   - The Morawetz/Rellich identity method of proof
4. Implications for Boundary Integral Equations
5. Implications for hp-BEM
6. Conclusions

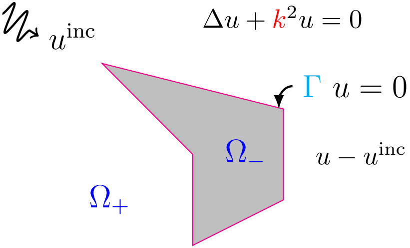
Integral Equations and k-Explicit Bounds



Assume throughout that Ω_- is bounded and Lipschitz.

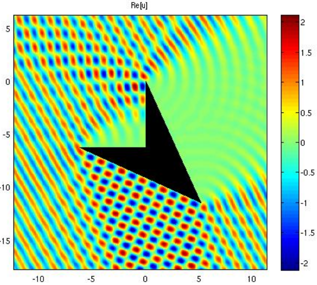
Integral Equations and k-Explicit Bounds

$$\Delta u + k^2 u = 0$$



$u - u^{\text{inc}}$ satisfies radiation condition

Assume throughout that Ω_- is bounded and Lipschitz. Plot of $\Re(u(x)) = U(x, 0)$:



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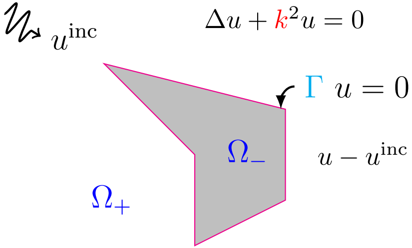

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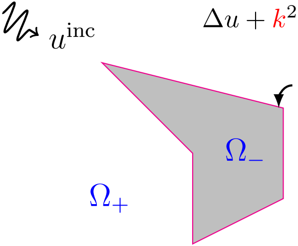
Assume throughout that Ω_- is bounded and Lipschitz.

Theorem (Green's Representation Theorem)

$$u(x) = u^{\text{inc}}(x) + \int_\Gamma \Phi(x,y)\partial_n^+ u(y)\, ds(y), \quad x \in \Omega_+.$$

where

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ (2D)}, \quad := \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$



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Γ $u = 0$

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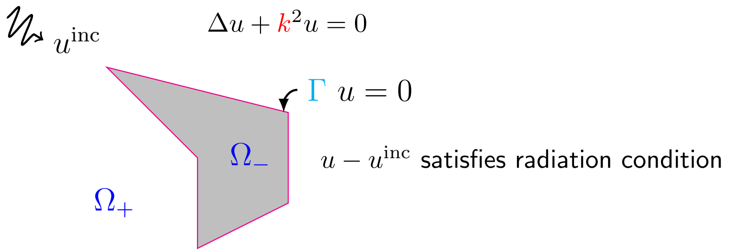

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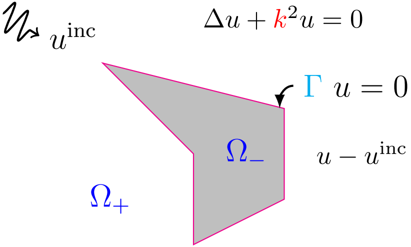
Taking a linear combination of Dirichlet (γ_+) and Neumann (∂_n^+) traces, we get the **boundary integral equation** (Burton & Miller 1971)

$$\frac{1}{2}\partial_n^+ u(x) + \int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)} + \mathrm{i}\eta\Phi(x,y) \right) \partial_n^+ u(y) ds(y) = f(x), \quad x \in \Gamma,$$

where

$$f := \partial_n^+ u^{\mathrm{inc}} + \mathrm{i}\eta\gamma_+ u^{\mathrm{inc}}.$$

$\Delta u + k^2 u = 0$



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in operator form

$$A_{k,\eta} \partial_n^+ u = f := \partial_n^+ u^{\text{inc}} + \mathrm{i}\eta\gamma_+ u^{\text{inc}}.$$

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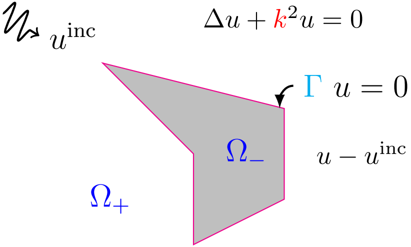

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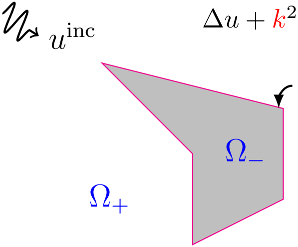
Theorem (Burton & Miller 1971, Mitrea 1996, C-W & Langdon 2007)

If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$.



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Γ $u = 0$

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The standard choice is $\eta = k$, and with this choice we have

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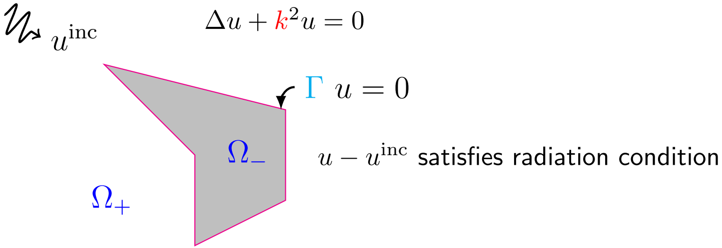
if Ω_- is **star-shaped** (C-W, Monk 2008) or C^∞ and **nontrapping** (Baskin, Spence, Wunsch 2016).



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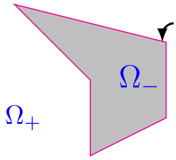
if Ω_- is **star-shaped** (C-W, Monk 2008) or C^∞ and **nontrapping** (Baskin, Spence, Wunsch 2016). But what if Ω_- is **trapping**?

A recipe for bounding $\|A_{k,k}^{-1}\|$ (C-W, Spence, Gibbs, Smyshlyaev 2017)

$\Delta u + k^2 u = f \in L^2(\Omega_+)$, compactly supported



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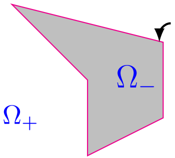

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where $\Omega_R := \{x \in \Omega_+ : |x| < R\}$.

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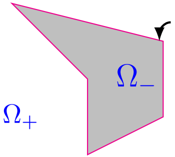

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Step 2 (DtN Map Bound). It follows that, if $f = 0$,

$$\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim c(k)\left(\|\nabla_\Gamma g\|_{L^2(\Gamma)} + k\|g\|_{L^2(\Gamma)}\right)$$

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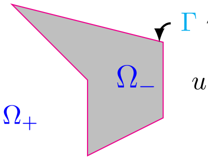

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Step 3 As (C-W, Graham, Langdon, Spence 2012)

$$A_{k,k}^{-1} = I - (P_{DtN}^+ - \mathrm{i}k)P_{ItD}^-$$

and P_{ItD}^- is bounded in Spence (2015), Baskin, Spence, Wunsch (2016), it follows that

$$\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim c(k)k^{1/2}$$

A recipe for bounding $\|A_{k,k}^{-1}\|$ (C-W, Spence, Gibbs, Smyshlyaev 2017)

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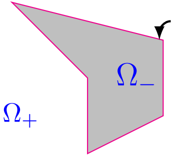


Γ $u = g \in H^1(\Gamma)$

u satisfies radiation condition

Step 1 (Resolvent Estimate). Show that, for every $R > 0$, if $g = 0$,

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_+)},$$

where $\Omega_R := \{x \in \Omega_+ : |x| < R\}$.

Step 2 (DtN Map Bound). It follows that, if $f = 0$,

$$\|\partial_n^+ u\|_{L^2(\Gamma)} \lesssim c(k)\left(\|\nabla_\Gamma g\|_{L^2(\Gamma)} + k\|g\|_{L^2(\Gamma)}\right)$$

Step 3 As (C-W, Graham, Langdon, Spence 2012)

$$A_{k,k}^{-1} = I - (P_{DtN}^+ - \mathrm{i}k)P_{ItD}^-$$

and P_{ItD}^- is bounded in Spence (2015), Baskin, Spence, Wunsch (2016), it follows that

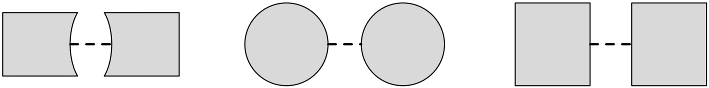
$$\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim c(k)$$

if each component of Ω_- is star-shaped or C^∞.

Recap of resolvent estimates for trapping obstacles

$$\|\nabla u\|_{L^2(\Omega_R)} + k\|u\|_{L^2(\Omega_R)} \lesssim c(k)\|f\|_{L^2(\Omega_R)}, \quad \text{for } k \geq k_0 > 0,$$

where $c(k) = 1$ for **nontrapping** obstacles, and



$c(k) = \exp(\alpha k)$
elliptic

$c(k) = \log(2 + k)$
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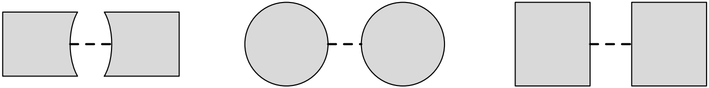

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Applying our general recipe

$$\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim c(k)$$

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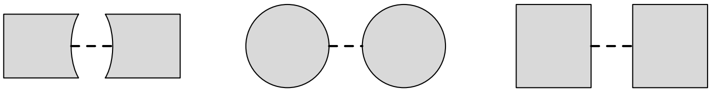

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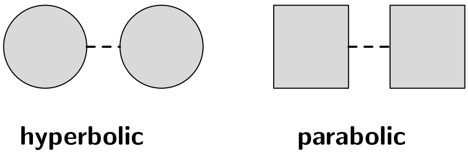
Applying our general recipe, for some $N \geq 0$,

$$\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim c(k) \lesssim k^N$$

in the **nontrapping** and **hyperbolic** and **parabolic trapping** cases.

Application to hp-BEM analysis



For these configurations $\exists N \geq 0$ s.t. $\|A_{k,k}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^N, \quad k \geq k_0 > 0.$

Corollary (Löhndorf, Melenk 2011)

Suppose Γ is analytic and $\mathcal{T}_h$ is a quasi-uniform triangulation with mesh size h. Then, given $k_0 > 0$, $\exists C_1, C_2, C_3$ such that, if $k \geq k_0$,

$$\frac{kh}{p} \leq C_1, \quad \textit{and} \quad p \geq C_2 \log(2+k),$$

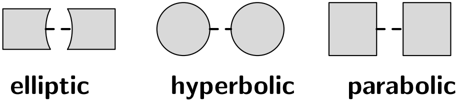
then the Galerkin hp-BEM solution $v_{hp} \in \mathcal{S}^p(\mathcal{T}_h)$ satisfies the quasi-optimal error estimate

$$\|v_{hp} - \partial_n^+ u\|_{L^2(\Gamma)} \leq C_3 \inf_{v \in \mathcal{S}^p(\mathcal{T}_h)} \|v - \partial_n^+ u\|_{L^2(\Gamma)}.$$

Conclusions

In this talk you have seen:

- All the resolvent estimates that exist for (Dirichlet) obstacles



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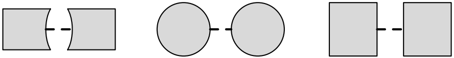

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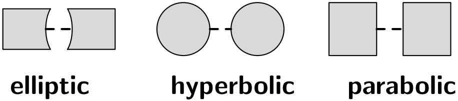

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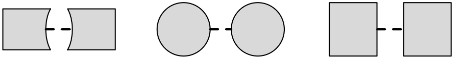

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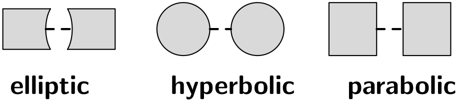

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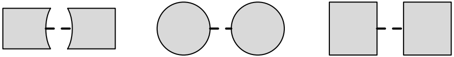

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More details see:

C-W, Spence, Gibbs, Smyshlyaev 2017, *High-frequency bounds for the Helmholtz equation under parabolic trapping and applications in numerical analysis*, arXiv:1708.08415