Do Galerkin methods converge for the classical 2nd kind boundary integral equations in polyhedra and Lipschitz domains?

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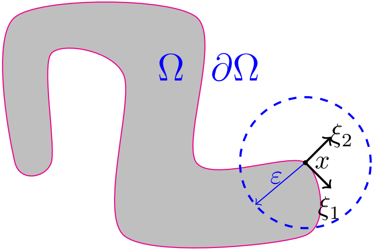
Joint work with:
Euan Spence (Bath)

Cardiff Seminar, November, 2022.

Overview

1. **Lipschitz domains** and an example we will meet later
2. **Potential theory, 2nd kind boundary integral equations, and a long-standing open question**
3. The **Hilbert space** theory of **Galerkin methods**
4. **Do all Galerkin BEMs, based on asymptotically dense subspace sequences and testing with L^2 inner products, converge for the standard 2nd kind BIEs on Lipschitz and polyhedral domains?**
5. **Some open questions**

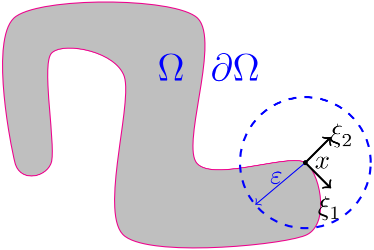


A bounded domain $\Omega \subset \mathbb{R}^2$ is **Lipschitz** if, in a neighbourhood of each point $x \in \partial\Omega$,

$$\partial\Omega \cap B_\epsilon(x) = \{(\xi_1, f(\xi_1)) : \xi_1 \in \mathbb{R}\} \cap B_\epsilon(x),$$

for some f that satisfies, for some $L > 0$ (the **Lipschitz constant**)

$$|f(s) - f(t)| \leq L|s - t|, \quad \text{for } s, t \in \mathbb{R}.$$



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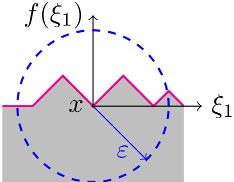
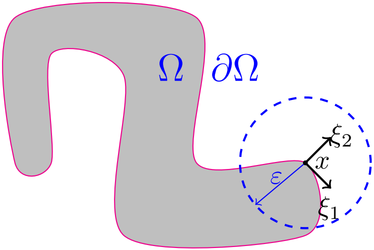
This **allows corners**, e.g. this f has $L = 1$ ...





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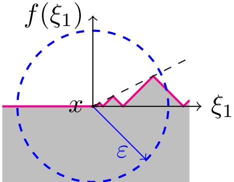
Indeed it allows **infinitely many corners**, e.g. this f also has $L = 1$ ...



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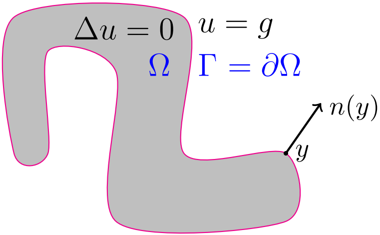
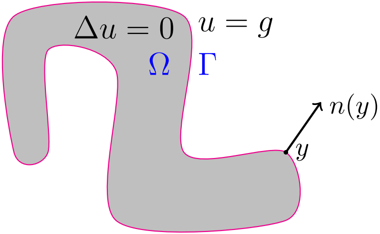
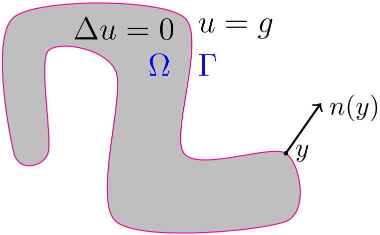


Assume that $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is **bounded** and **Lipschitz**, and $g \in L^2(\Gamma)$.



BVP: Find $u \in C^2(\Omega)$ such that $\Delta u = 0$ in Ω and $u = g \in L^2(\Gamma)$ on Γ.



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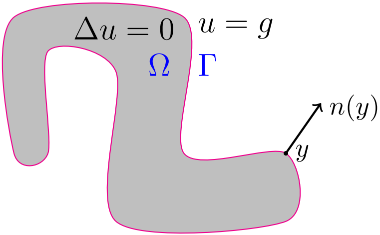
Define the **fundamental solution**

$$G(x,y) := \begin{cases} -\frac{1}{\pi}\log|x-y|, & d=2, \\ (2\pi|x-y|)^{-1}, & d=3, \end{cases}$$

Look for a solution as the **double-layer potential** with density $\phi \in L^2(\Gamma)$:

$$\begin{aligned} u(x) &= \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)}\phi(y)\,\mathrm{d}s(y) \\ &= \frac{1}{2^{d-2}\pi}\int_\Gamma \frac{(x-y)\cdot n(y)}{|x-y|^d}\phi(y)\,\mathrm{d}s(y), \end{aligned}$$

for $x \in \Omega$.



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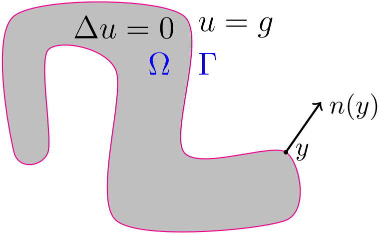
for $x \in \Omega$. This idea (with $\phi \in C(\Gamma)$) dates back to Gauss.



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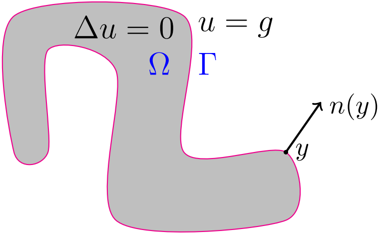
$$u(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y) \, \mathrm{d}s(y), \quad x \in \Omega.$$

This satisfies the BVP iff ϕ satisfies the **boundary integral equation (BIE)**

$$\phi(x) - \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y) \, \mathrm{d}s(y) = -g(x), \quad x \in \Gamma,$$



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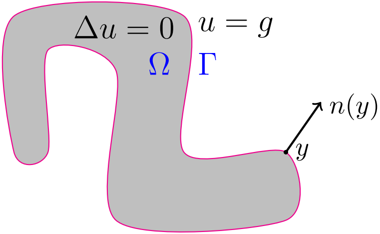
in operator form

$$\phi - D\phi = -g \quad \text{or } A\phi = -g,$$

where $A = I - D$, I is the identity operator, and D is the **double-layer potential operator** given by

$$D\phi(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y) \, \mathrm{d}s(y), \quad x \in \Gamma, \; \phi \in L^2(\Gamma).$$



The double-layer potential satisfies the BVP iff ϕ satisfies the **BIE** in operator form

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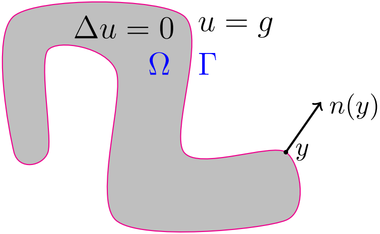

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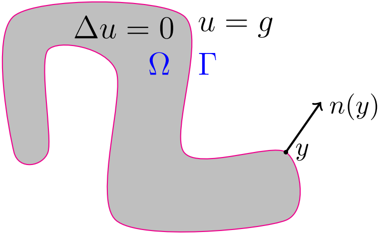
$$\phi \approx \phi_N \in V_N,$$

where

$$(A\phi_N, \psi_N) = -(g, \psi_N), \quad \forall \psi_N \in V_N, \quad \text{and } (u, v) := \int_\Gamma u\bar{v}\,\mathrm{d}s.$$



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Long-standing open problem. "For a general Lipschitz boundary Γ, however, stability and convergence of Galerkin's method in $L^2(\Gamma)$ is not yet known." **Wendland** (2009)

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H is a **complex Hilbert space** with inner product (u,v) and norm $\|u\| = \sqrt{(u,u)}$, e.g.

$$H = L^2(\Gamma), \quad (u,v) = \int_\Gamma u\overline{v}\,\mathrm{d}s, \quad \|u\|^2 = \int_\Gamma |u|^2\,\mathrm{d}s.$$

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A is a **bounded linear operator** on H if

$$A(\lambda u) = \lambda\, Au, \quad A(u + v) = Au + Av, \quad \forall \lambda \in \mathbb{C},\ u, v \in H,$$

and, for some $C \geq 0$,

$$\|Au\| \leq C\|u\|, \quad \forall u \in H.$$

The **norm** of A is

$$\|A\| := \sup_{u \in H\setminus\{0\}} \frac{\|Au\|}{\|u\|}.$$

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A is **compact** if, for some sequence of finite rank operators $A_1, A_2, \ldots$, it holds that $\|A - A_n\| \to 0$ as $n \to \infty$.

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Suppose that A is a **bounded linear operator** on H.

A is **coercive** if, for some $\gamma > 0$,

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So $A = I - B$ is coercive if $\|B\| < 1$, with $\gamma = 1 - \|B\|$.

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So $A = I - B$ is coercive if $\|B\| < 1$, with $\gamma = 1 - \|B\|$.

Indeed A is coercive iff $A = \theta(I - B)$ with $\theta \in \mathbb{C} \setminus 0$ and $\|B\| < 1$.

Suppose that A is a **bounded linear operator** on H.

The Galerkin method. Pick a sequence $V = (V_1, V_2, ...)$ of finite-dimensional subspaces of H, and seek $u_N \in V_N$ such that

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In the case that A is invertible, we will say that the **Galerkin method is convergent for the sequence** V if, for every $g \in H$, (G) has a unique solution for all sufficiently large N and $u_N \to u := A^{-1}g$ as $N \to \infty$.

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We will say that V **is asymptotically dense in** H if, for every $u \in H$,

$$\inf_{v_N \in V_N} \|u - v_N\| \to 0 \quad \text{as} \quad N \to \infty.$$

It is easy to see that a **necessary condition** for the convergence of the Galerkin method is that V is asymptotically dense in H.

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The Key Abstract Theorem. (Markus, 1974). If A is invertible then the following statements are equivalent:

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- $A = A_0 + K$ where A_0 is **coercive** and K is **compact**.

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The above implies that, if A is not coercive + compact, then there exists **at least one** asymptotically dense sequence $V = (V_1, V_2, ...)$ for which the Galerkin method does not converge.

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Theorem. (C-W, Spence 2022) If A is not coercive + compact then, for every asymptotically dense $V = (V_1, V_2, ...)$, there exists a sequence $V^* = (V_1^*, V_2^*, ...)$ for which the Galerkin method does not converge which is **sandwiched by** V, meaning that, for each N,

$$V_N \subset V_N^* \subset V_{M_N}, \quad \text{for some } M_N \geq N.$$

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N.B. $V_N \subset V_N^*$ implies that V^* is also asymptotically dense.

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What is known about the double-layer potential operator D and $A = I - D$ when Ω is Lipschitz? Remember the BIE in operator form is $A\phi = -g$.

- A is a bounded linear operator on $L^2(\Gamma)$ if Ω is a bounded Lipschitz domain (Coifman, McIntosh, Meyer *Ann. Math.* 1982)
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- The same holds if Γ is Lipschitz with small Lipschitz constant (I. Mitrea *J. Fourier Anal. Appl.* 1999, C-W, Spence *Numer. Math.* 2022)

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- The same holds if Γ is Lipschitz with small Lipschitz constant (I. Mitrea *J. Fourier Anal. Appl.* 1999, C-W, Spence *Numer. Math.* 2022)
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The answer is **NO** in each case (C-W & Spence, *Numer. Math.* 2022).

The Galerkin method. Pick a sequence $V = (V_1, V_2, ...)$ of finite-dimensional subspaces of H, and seek $u_N \in V_N$ such that

$$(Au_N, v) = (g, v), \quad \forall v \in V_N.$$

The Key Abstract Theorem extended.

If A is invertible then the following statements are equivalent:

- The Galerkin method converges for every V that is asymptotically dense in H.
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Here $W_{\mathrm{ess}}(A)$ denotes the **essential numerical range** of A, defined by

$$W_{\mathrm{ess}}(A) := \bigcap_{K \text{ compact}} W(A + K),$$

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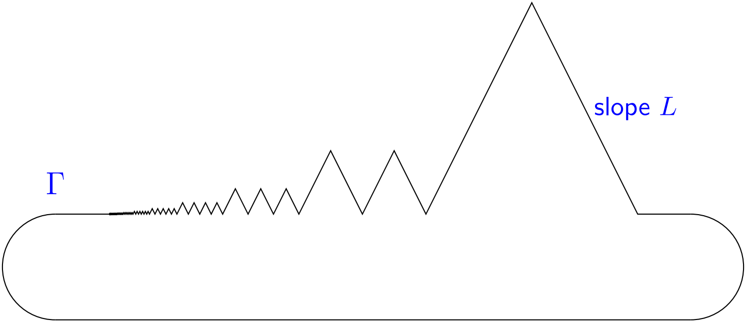
Key question: If $A = I - D$ and D is the double-layer potential operator, is $0 \in W_{\text{ess}}(A)$? Equivalently, is $1 \in W_{\text{ess}}(D)$?

Theorem. (C-W, Spence 2022) If Γ is the boundary of the Lipschitz domain shown below with Lipschitz constant L, then

$$W_{\mathrm{ess}}(D) \supset \{z \in \mathbb{C} : |z| \leq L/2\}.$$

Thus, if $L \geq 2$, then $1 \in W_{\mathrm{ess}}(D)$, so that $A = I - D$ is not coercive + compact.



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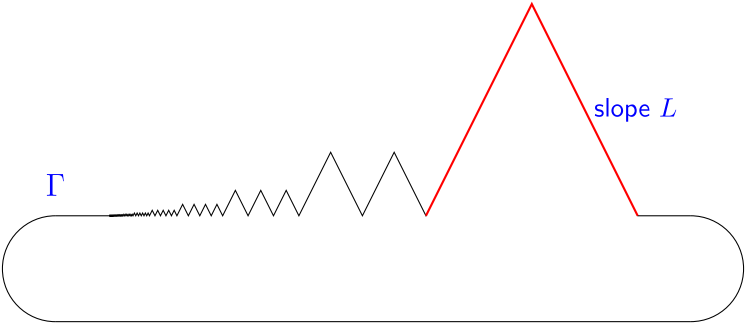

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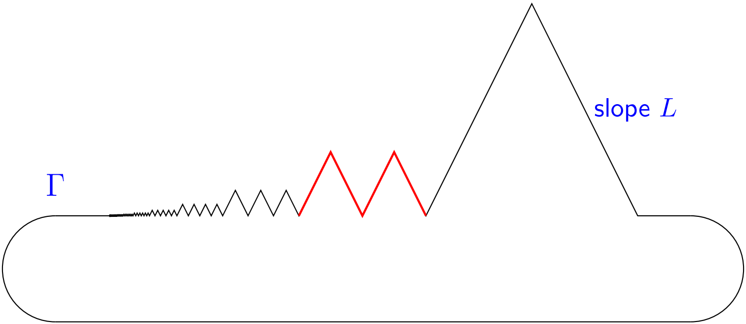

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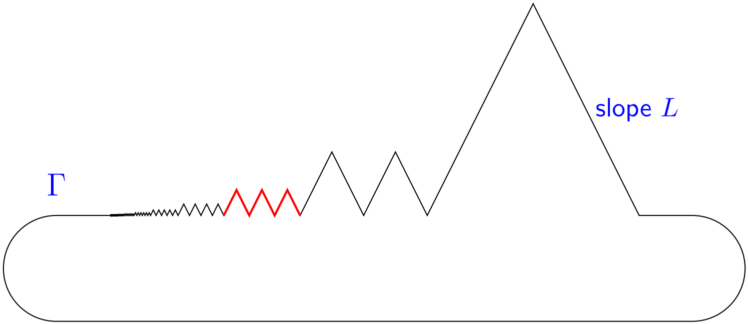

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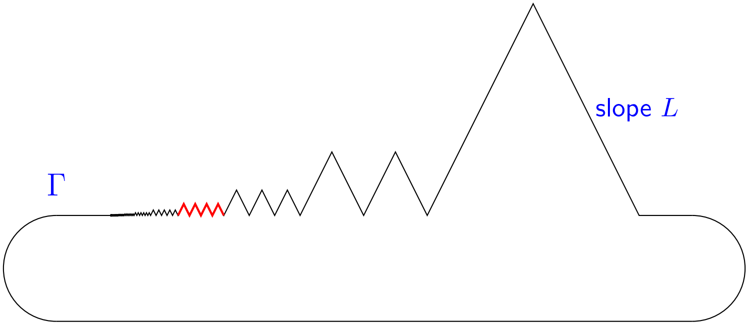

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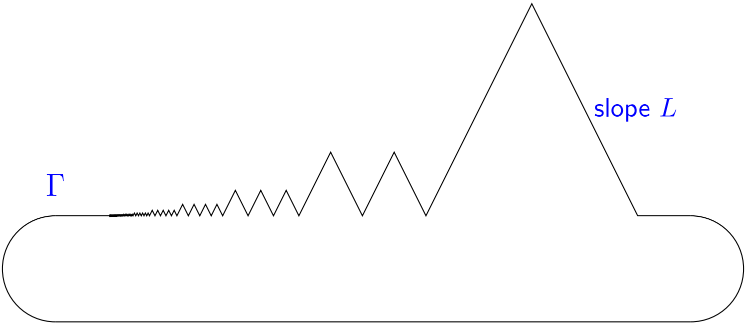

How is this proved?

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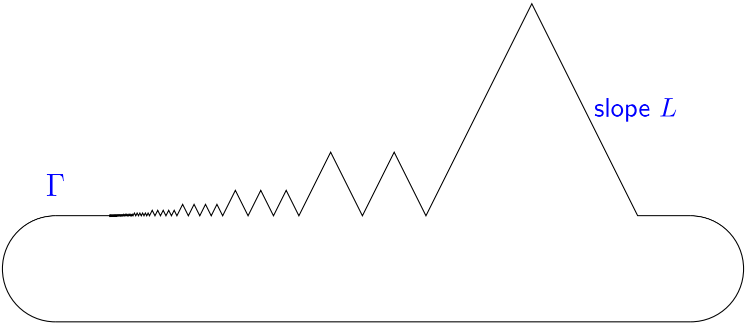
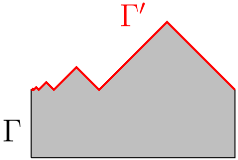


How is this proved? By three simple lemmas and a calculation ...

Three simple lemmas.

Lemma A. If $\Gamma' \subset \Gamma$ and D' is the DLP operator on Γ', then

$$W(D') \subset W(D).$$



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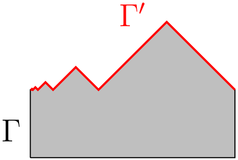
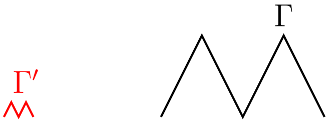


Lemma B. If Γ' and Γ are similar and D' is the DLP operator on Γ', then

$$W(D') = W(D).$$



Lemma C. If $\Gamma_1 \subset \Gamma_2 \subset \ldots \Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$, and D_j denotes the DLP on Γ_j, then

$$W(D) = \overline{\bigcup_{j=1}^{\infty} W(D_j)}.$$

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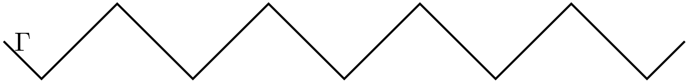

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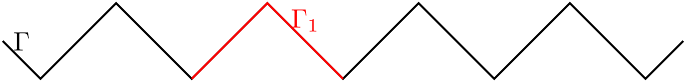

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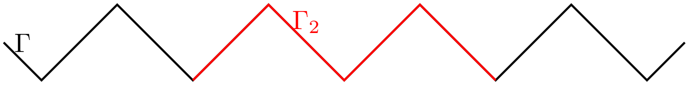

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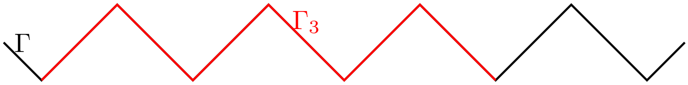

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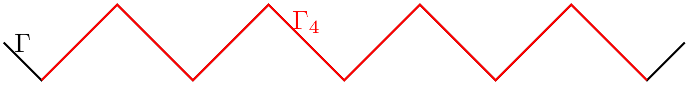
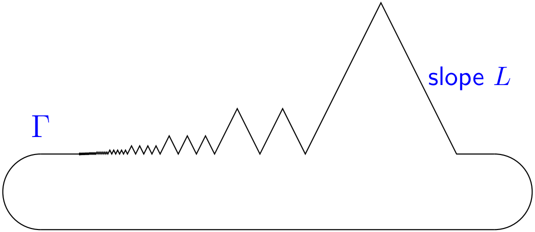


What can we say about $W(D)$ for the DLP operator D on this Γ?



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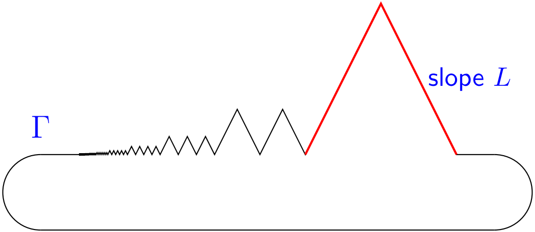

By **Lemma A**, $W(D) \supset W(D')$ where D' is the DLP operator on each of these Γ' in **red**.

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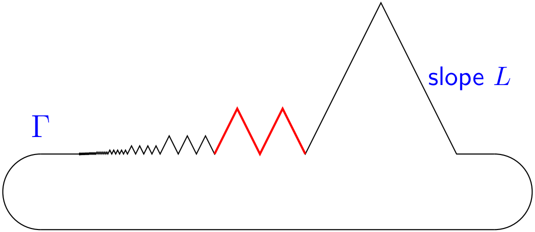

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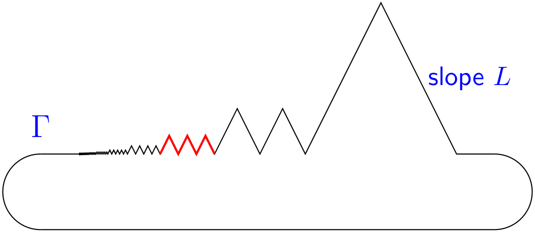

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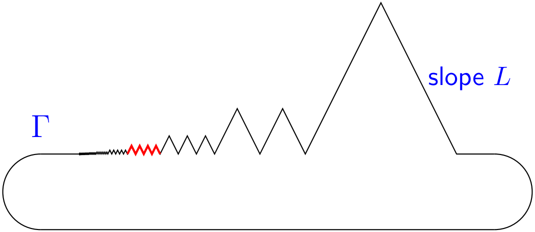

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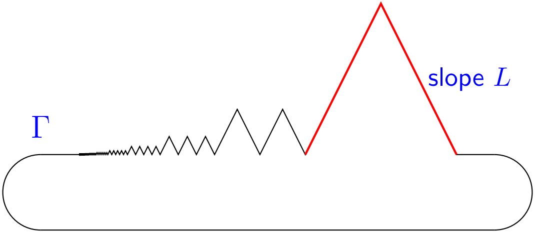

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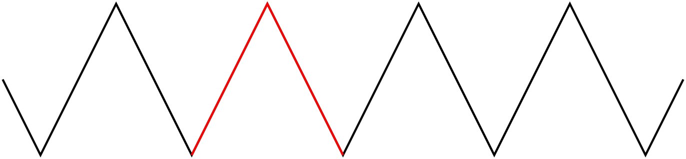

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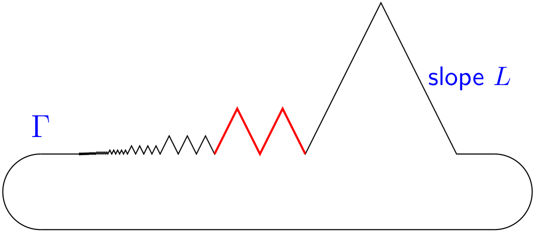

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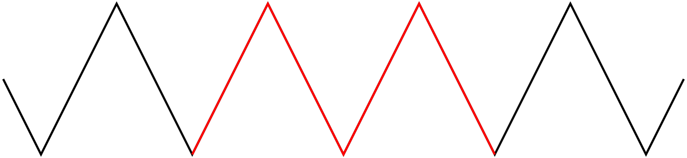

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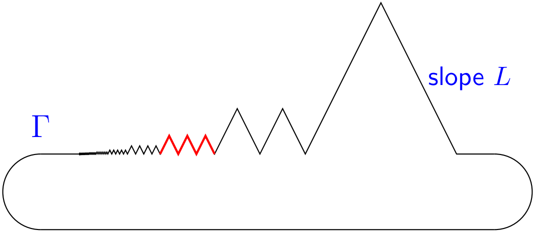

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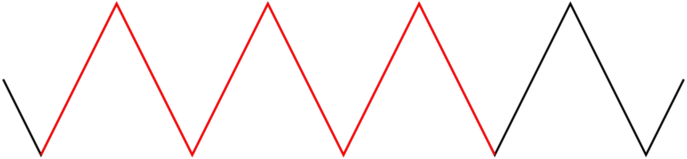

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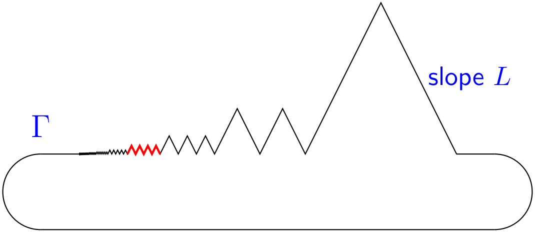

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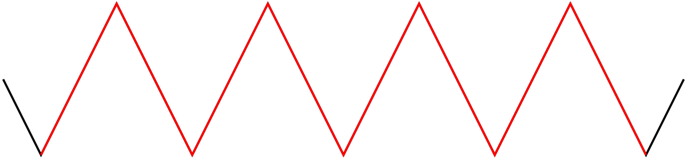

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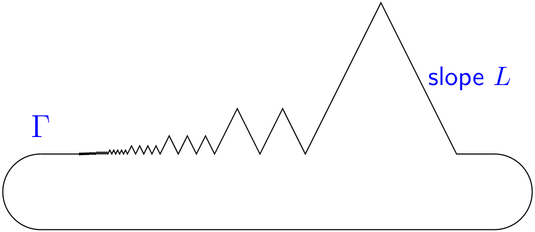

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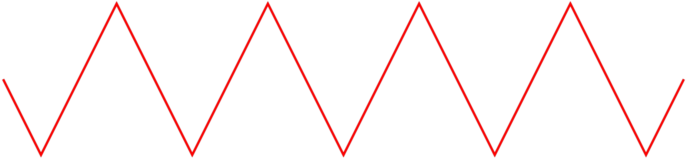
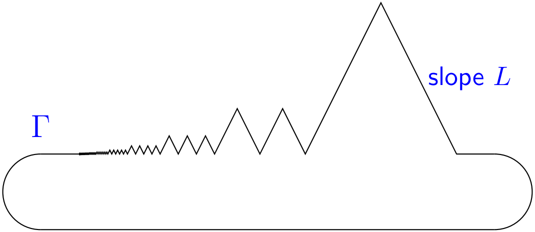


So, by **Lemma C**, also $W(D) \supset W(D^\dagger)$ where $D^\dagger$ is the DLP operator on the **infinite sawtooth**.

What can we say about $W(D)$ for the DLP operator D on this Γ?



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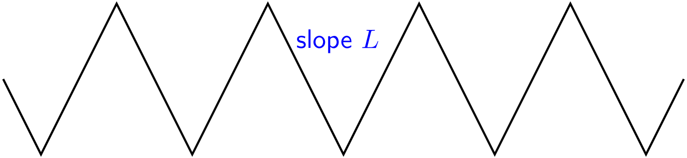

So, by **Lemma C**, also $W(D) \supset W(D^\dagger)$ where $D^\dagger$ is the DLP operator on the **infinite sawtooth**. And $W(D^\dagger)$ (by some explicit calculations – see later) contains $\{z \in \mathbb{C} : |z| \leq L/2\}$.

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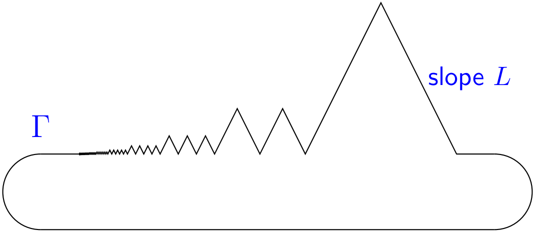

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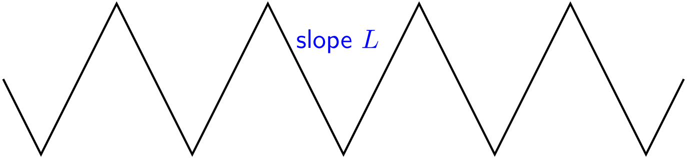

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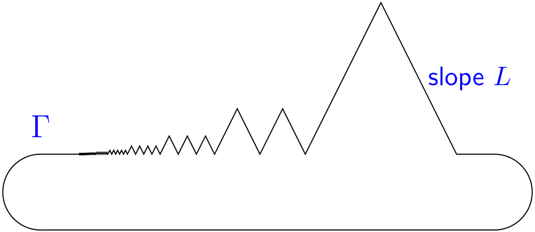

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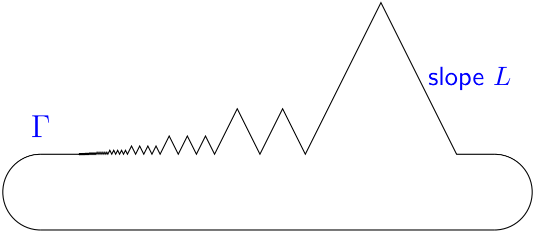

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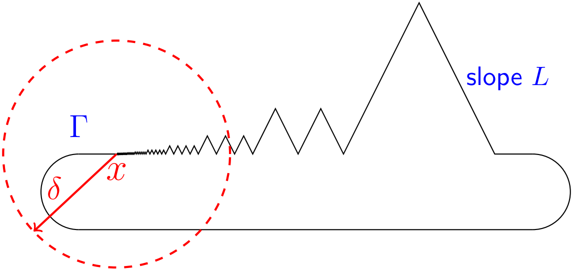

Localisation Lemma. (C-W, Spence 2022, cf. I. Mitrea, 1999)

$$W_{\mathrm{ess}}(D) \supseteq \bigcap_{\delta>0} W\left(D_{x,\delta}\right), \quad \forall x \in \Gamma,$$

where $D_{x,\delta}$ is the DLP operator on $\Gamma \cap B_\delta(x)$.

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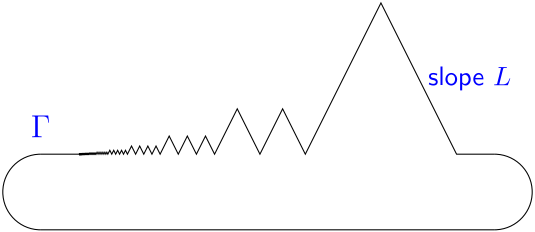
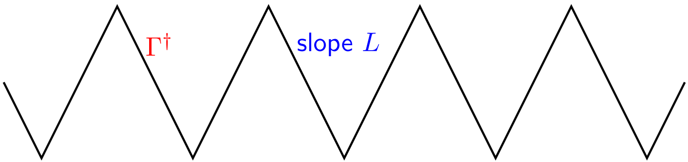


The DLP operator $D^\dagger$ on the sawtooth graph $\Gamma^\dagger$



Theorem. Let $D^\dagger$ be the DLP operator on the **infinite sawtooth** $\Gamma^\dagger$ with **slope** L. Then, as an operator on $L^2(\Gamma^\dagger)$,

$$W(D^\dagger) \supset \{z \in \mathbb{C} : |z| \leq L/2\} \quad \text{and} \quad \|D^\dagger\| \geq L.$$

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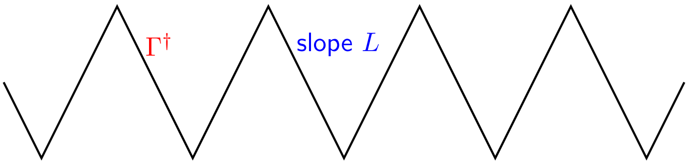

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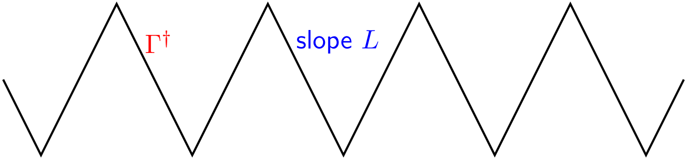

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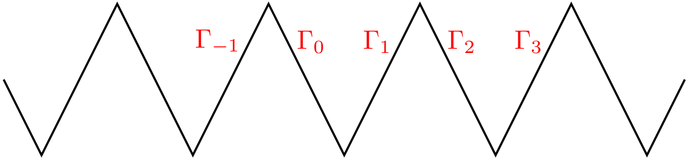
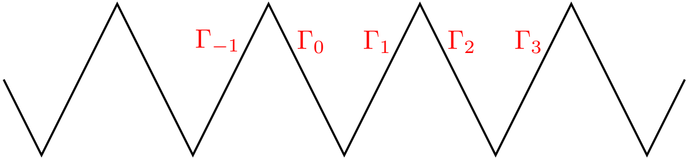
$$W(D^\dagger) \supset W(PD^\dagger|_{V_*}) \quad \text{and} \quad \|D^\dagger\| \geq \|PD^\dagger|_{V_*}\|.$$



Proof continued ... Moreover, for $\phi \in V_*$,

$$\left(PD^\dagger\phi\right)\big|_{\Gamma_m} = \sum_{n=-\infty}^{\infty} a_{m-n}\phi\big|_{\Gamma_n}(-1)^n, \quad \text{where} \quad a_n := \operatorname{sgn}(n)\ \left|(D^\dagger\chi_0, \chi_n)\right|,$$

and χ_n is the normalised characteristic function of Γ_n.



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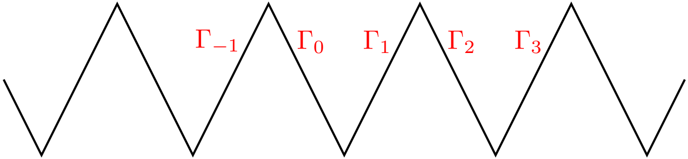

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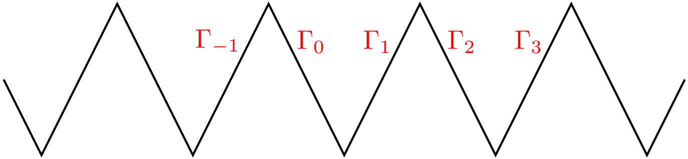

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Thus

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In conclusion we have proved ...

Theorem. (C-W, Spence 2022) If Γ is the boundary of the Lipschitz domain shown below with Lipschitz constant L, then

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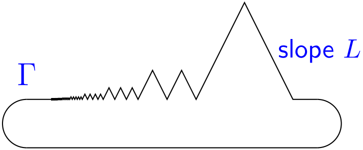

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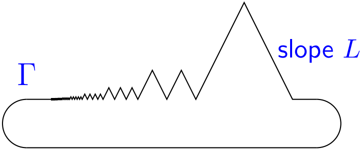

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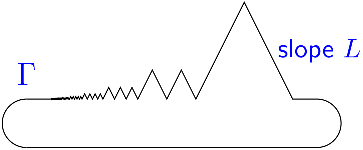

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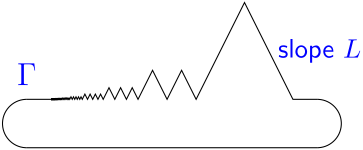

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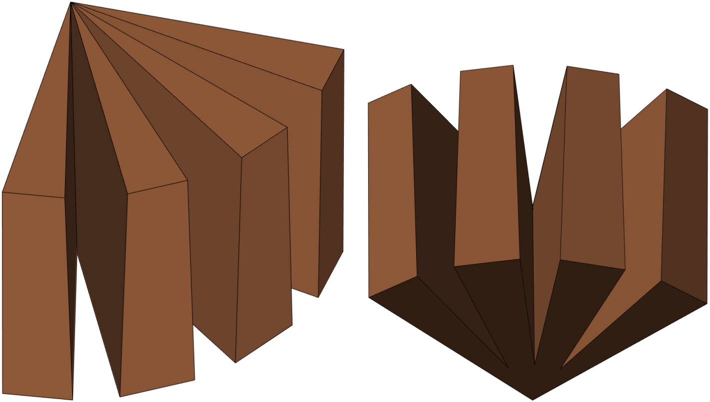
Choose V to be any asymptotically dense sequence of **BEM** spaces. Then V^* is a BEM space sequence ($V_N^* \subset V_{M_N}$) that is asymptotically dense ($V_N \subset V_N^*$) for which **the Galerkin method does not converge**.

3D Polyhedra for which $A = I - D$ is not coercive + compact.

The "open book" polyhedron with four pages and opening angle $\theta = \pi/4$.



Some Open Questions

- Are there Galerkin BEMs that provably converge for all Lipschitz domains, or at least for all polyhedra (cf. Elschner 1992a, b, 1995) ?
- Conversely, concrete examples of Galerkin BEMs that are not convergent?

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$$w_{\mathrm{ess}}(D) := \sup\{|z| : z \in W_{\mathrm{ess}}(D)\} \geq L/2,$$

but Kenig (1994) has conjectured that

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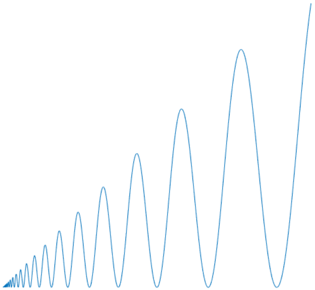

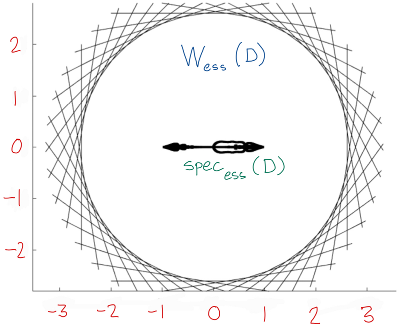

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- We have seen that $I - D$ is not always coercive + compact. But are there alternative 2nd kind formulations that are coercive + compact for every Lipschitz Ω? Yes, in fact even coercive (C-W, Spence, arXiv:2210.02432, 2022).

For more details see the open access paper ...

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Numerische Mathematik

Coercivity, essential norms, and the Galerkin method for second-kind integral equations on polyhedral and Lipschitz domains

S. N. Chandler-Wilde[1] · E. A. Spence[2]

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Abstract

It is well known that, with a particular choice of norm, the classical double-layer potential operator D has essential norm $< 1/2$ as an operator on the natural trace space $H^{1/2}(\Gamma)$ whenever Γ is the boundary of a bounded Lipschitz domain. This implies, for the standard second-kind boundary integral equations for the interior and exterior Dirichlet and Neumann problems in potential theory, convergence of the Galerkin method in $H^{1/2}(\Gamma)$ for any sequence of finite-dimensional subspaces $(\mathcal{H}_N)_{N=1}^{\infty}$ that is asymptotically dense in $H^{1/2}(\Gamma)$. Long-standing open questions are whether the essential norm is also $< 1/2$ for D as an operator on $L^2(\Gamma)$ for all Lipschitz Γ in 2-d; or whether, for all Lipschitz Γ in 2-d and 3-d, or at least for the smaller class of Lipschitz polyhedra in 3-d, the weaker condition holds that the operators $\pm\frac{1}{2}I + D$ are compact perturbations of coercive operators—this a necessary and sufficient condition for the