

Computing the Spectra and Pseudospectra of Band-Dominated Operators

Simon Chandler-Wilde

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This talk is based on joint work, see
<https://arxiv.org/abs/2401.03984>, with

- Marko Lindner, TU Hamburg, Germany
- Ratchanikorn Chonchaiya, King Mongkut's University of Technology, Thailand

and supported by Marie Curie Grants of the European Union.

What is this talk about?

Question. Given a bounded linear operator A on a Hilbert space E , can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\text{Spec } A \subset U_n$ for each n ;
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Answer. A qualified **yes**, if the matrix representation of A , with respect to some orthonormal sequence, is banded or band-dominated.

Novelty? We know how to construct U_n satisfying (iii) with $U_n \rightarrow \text{Spec}_\varepsilon A$, the ε -pseudospectrum, for band-dominated A (see Hansen 2011, Ben-Artzi, Colbrook, Hansen, Nevanlinna, Seidel 2015, 2020). But not known how to achieve (ii) and (iii).

Bounded linear operators between Hilbert spaces

E is a **complex Hilbert space** with inner product (x, y) and norm $\|x\| = \sqrt{(x, x)}$, e.g.

$$E = \ell^2 := \ell^2(\mathbb{Z}), \quad (x, y) = \sum_{j \in \mathbb{Z}} x_j \bar{y}_j, \quad \|x\|^2 = \sum_{j \in \mathbb{Z}} |x_j|^2.$$

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If E, Y are Hilbert spaces, A is a **bounded linear operator** from E to Y , in symbols $A \in L(E, Y)$, if

$$A(\lambda x) = \lambda Ax, \quad A(x + y) = Ax + Ay, \quad \forall \lambda \in \mathbb{C}, x, y \in E,$$

and, for some $C \geq 0$,

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For $A \in L(E, Y)$ the **norm** and **lower norm** of A are

$$\|A\| := \sup_{x \in E \setminus \{0\}} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \nu(A) := \inf_{x \in E \setminus \{0\}} \frac{\|Ax\|}{\|x\|}.$$

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We call $A \in L(E) := L(E, E)$

- **self-adjoint** if $A^* = A$
- **normal** if $AA^* = A^*A$

Bounded linear operators between Hilbert spaces

$A \in L(E) := L(E, E)$ is said to be **invertible** if it is bijective, in which case there exists $A^{-1} \in L(E)$ such that $AA^{-1} = A^{-1}A = I$.

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With the conventions that $\|A^{-1}\| := \infty$ if A is not invertible and $1/\infty := 0$,

$$\mu(A) = 1/\|A^{-1}\|, \quad \text{for all } A \in L(Y).$$

Spectrum and Pseudospectrum

For $A \in L(E)$ the **spectrum** of A is

$$\operatorname{Spec} A := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\} = \{\lambda \in \mathbb{C} : \mu(A - \lambda I) = 0\}.$$

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where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

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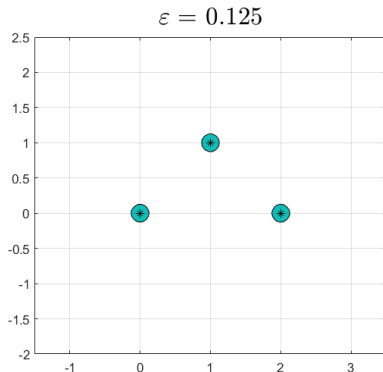
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$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+i \end{bmatrix}$$



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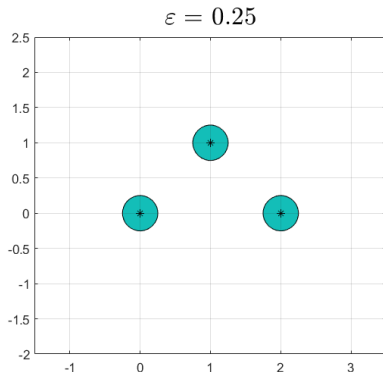
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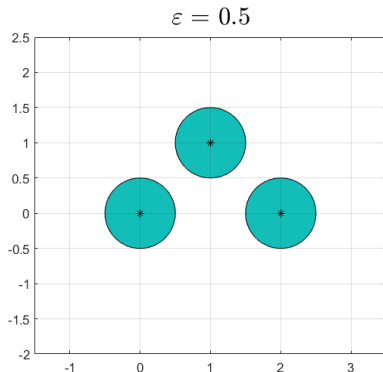
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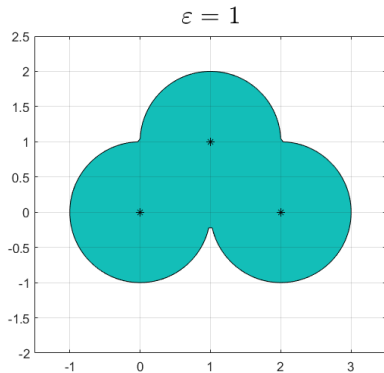
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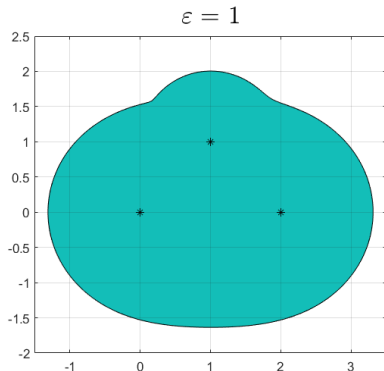
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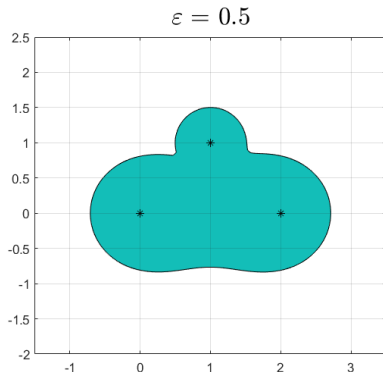
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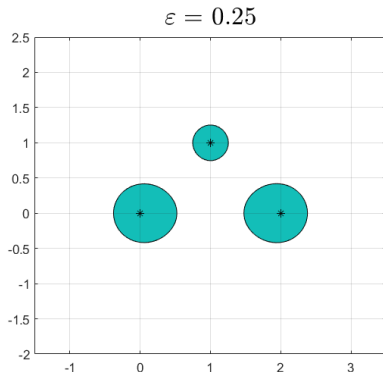
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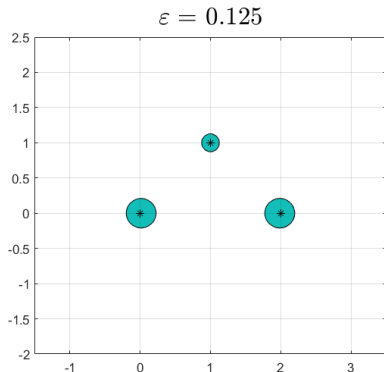
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Corollary. If $\varepsilon_1 > \varepsilon_2 > \dots > 0$, in which case $\varepsilon_n \rightarrow \varepsilon \geq 0$ as $n \rightarrow \infty$, then

$$\text{Spec}_{\varepsilon_n} A \rightarrow \text{Spec}_\varepsilon A \quad \text{N.B. } \text{Spec}_0 A := \text{Spec } A.$$

Matrix representation of A

Suppose $(e_j)_{j \in \mathbb{Z}}$ is an orthonormal basis for a separable Hilbert space E and $A \in L(E)$. Then the **matrix representation** of A is $[A] = [a_{ij}]_{i,j \in \mathbb{Z}}$, where

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and $\text{Spec } A = \text{Spec } [A]$, $\text{Spec}_\varepsilon A = \text{Spec}_\varepsilon [A]$, $\varepsilon > 0$, where $[A] \in L(\ell^2)$ is defined by

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We will say that $[A]$ is **band-dominated** if there exists a sequence $(A_n) \subset L(E)$ such that each $[A_n]$ is banded and $\|A - A_n\| \rightarrow 0$ as $n \rightarrow \infty$.

The tridiagonal case

Let's consider first bi-infinite matrices of the form

$$A = \begin{pmatrix} \ddots & & \ddots & & & & \\ & \ddots & \beta_{-2} & \gamma_{-1} & & & \\ & & \alpha_{-2} & \beta_{-1} & \gamma_0 & & \\ & & & \alpha_{-1} & \beta_0 & \gamma_1 & \\ & & & & \alpha_0 & \beta_1 & \gamma_2 \\ & & & & & \alpha_1 & \beta_2 & \ddots \\ & & & & & & \ddots & \ddots \end{pmatrix},$$

where $\alpha = (\alpha_i)$, $\beta = (\beta_i)$ and $\gamma = (\gamma_i)$ are bounded sequences of complex numbers.

Inclusion sets for $\text{Spec}_\varepsilon A$, $\varepsilon \geq 0$.

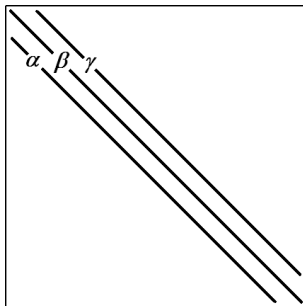
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Task

Compute **inclusion sets for spectrum and pseudospectra** of $A \in L(\ell^2) = L(\ell^2(\mathbb{Z}))$.

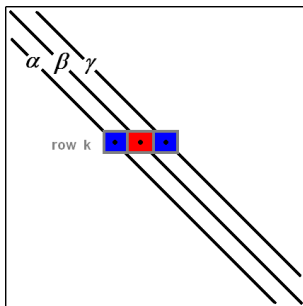
Inspiration: Gershgorin discs

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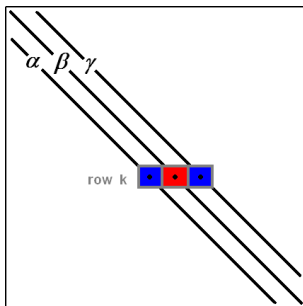


For every row k , consider the **Gershgorin disc** with

$$\text{center at } a_{k,k} \text{ and radius } |a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$$

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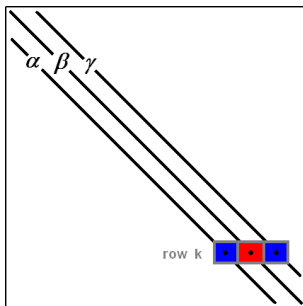


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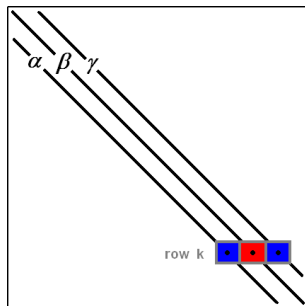


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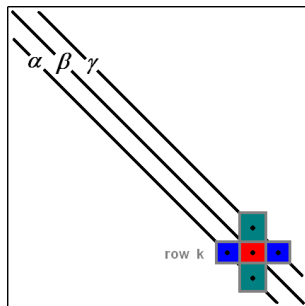
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center at $a_{k,k}$ and radius $|a_{k,k-1}| + |a_{k,k+1}| \leq \|\alpha\|_\infty + \|\gamma\|_\infty$

$$\text{Spec } A \subset \overline{\bigcup_{k \in \mathbb{Z}} (a_{k,k} + (\|\alpha\|_\infty + \|\gamma\|_\infty)\mathbb{D})}.$$

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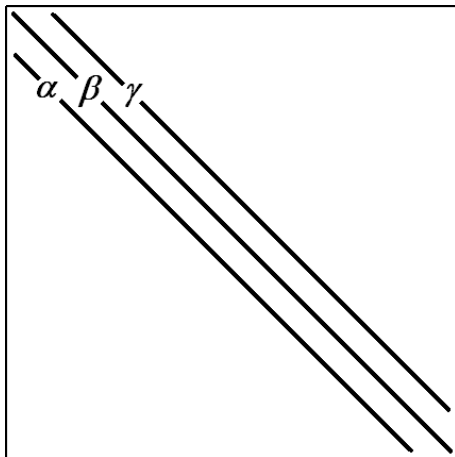
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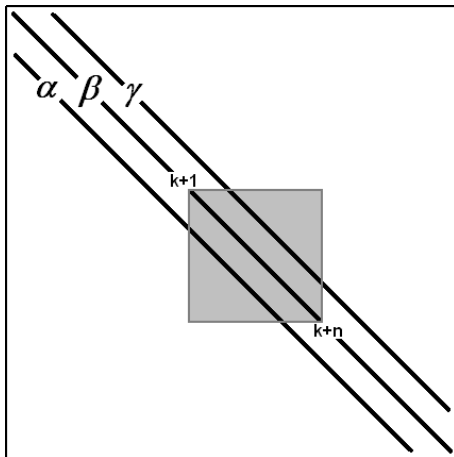
Our new strategy

Look at (pseudo)spectra of the **finite principal submatrices** of A :



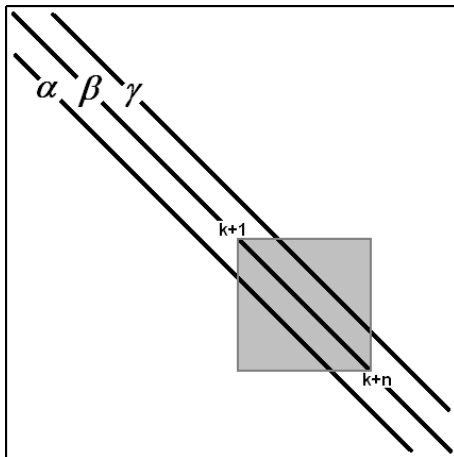
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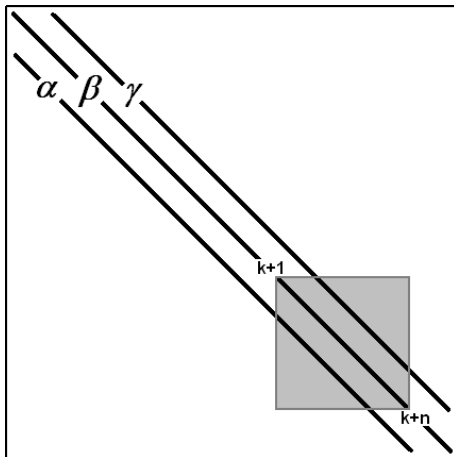
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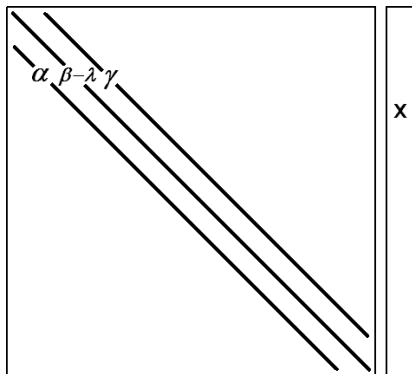


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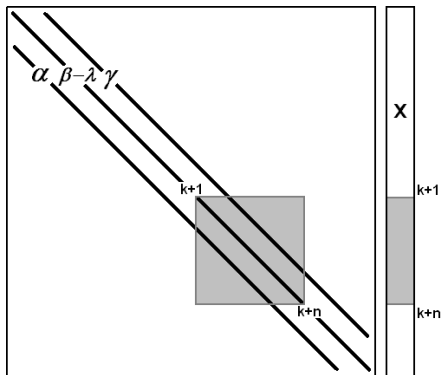
Let $\lambda \in \text{Spec}_\varepsilon A$ and let $x \in \ell^2$ be a corresponding pseudomode.



$$\|(A - \lambda I)x\| \leq \varepsilon \|x\|$$

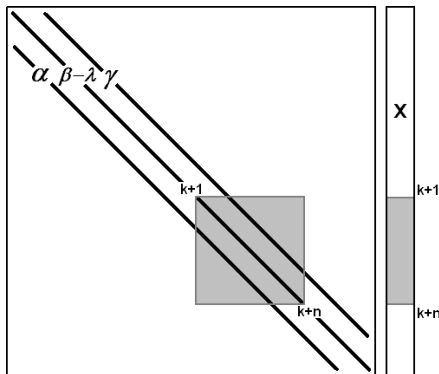
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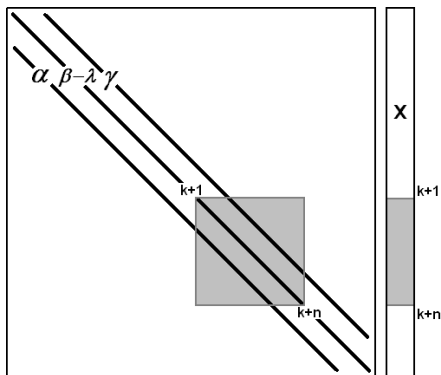
$$\|(A - \lambda I)x\| \leq \varepsilon \|x\|$$

Claim: $\exists k \in \mathbb{Z}$:

$$\begin{aligned} \|(A_{n,k} - \lambda I_n) x_{n,k}\| \\ \leq (\varepsilon + \varepsilon_n) \|x_{n,k}\| \end{aligned}$$

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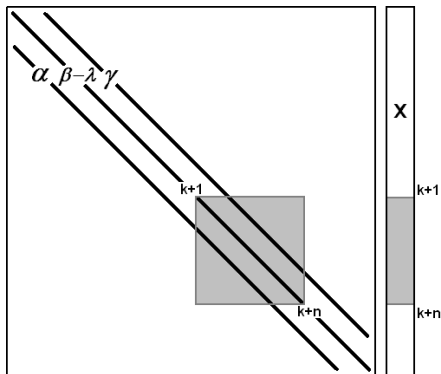
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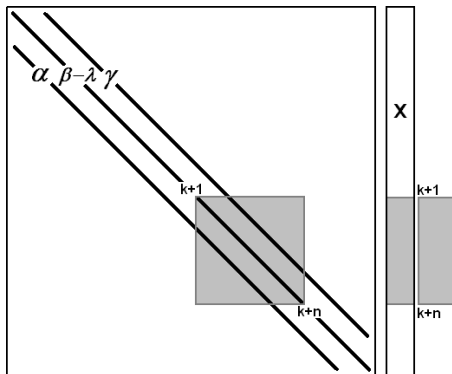
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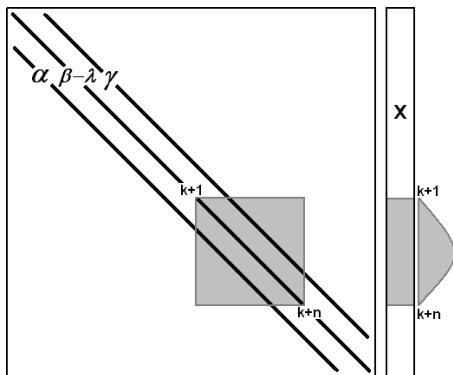
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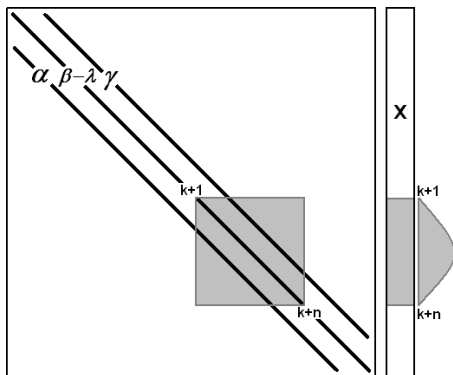
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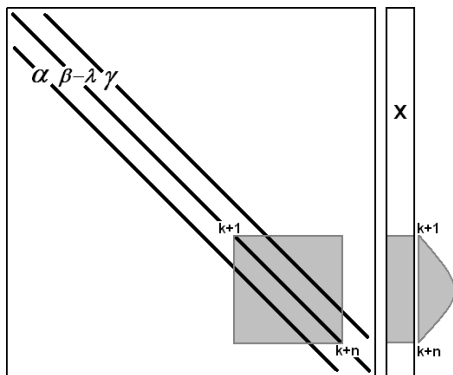
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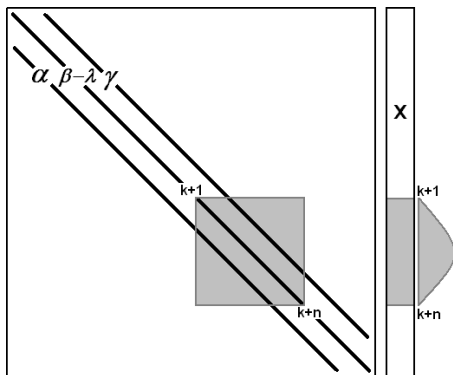
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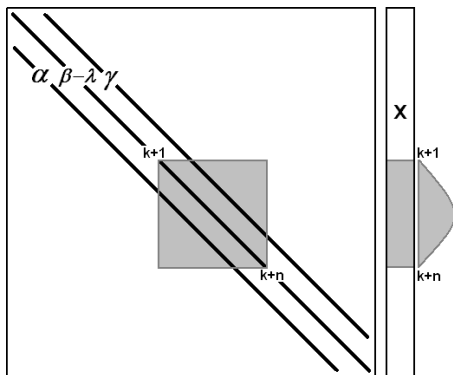
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So we get

Inclusion Set

$$\operatorname{Spec}_{\varepsilon} A \subset \overline{\bigcup_{k \in \mathbb{Z}} \operatorname{Spec}_{\varepsilon + \varepsilon_n} A_{n,k}}, \quad \varepsilon \geq 0,$$

where

$$\varepsilon_n \leq 2 \sin \left(\frac{\pi}{2(n+2)} \right) (\|\alpha\|_{\infty} + \|\gamma\|_{\infty}),$$

so $\varepsilon_n = O(n^{-1})$ as $n \rightarrow \infty$.

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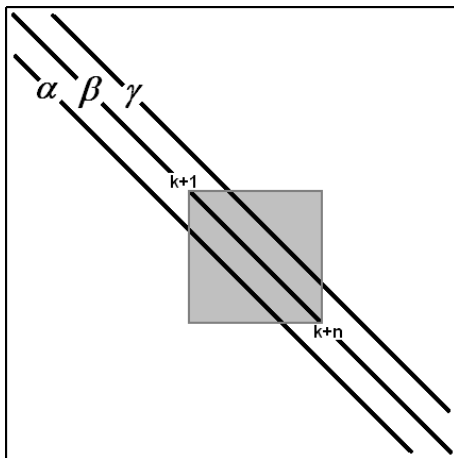
$$\varepsilon_n \leq 2 \sin \left(\frac{\pi}{2(n+2)} \right) (\|\alpha\|_{\infty} + \|\gamma\|_{\infty}),$$

so $\varepsilon_n = O(n^{-1})$ as $n \rightarrow \infty$. Putting $n = 1$ and $\varepsilon = 0$ we recover Gershgorin:

$$\operatorname{Spec} A \subset \overline{\bigcup_{k \in \mathbb{Z}} \operatorname{Spec}_{\varepsilon_1} A_{1,k}} = \overline{\bigcup_{k \in \mathbb{Z}} (a_{k,k} + (\|\alpha\|_{\infty} + \|\gamma\|_{\infty})\mathbb{D})}.$$

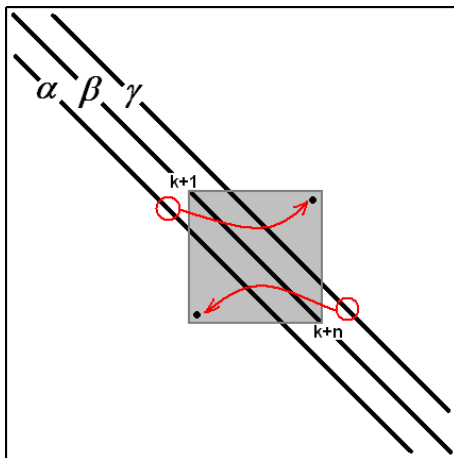
π method: **periodised** finite principal submatrices

If the finite submatrices $A_{n,k}$ are “periodised” (cf. Colbrook 2020, which uses **single large periodised finite section**)



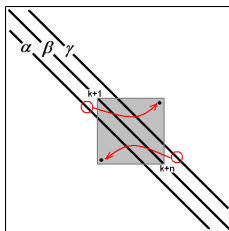
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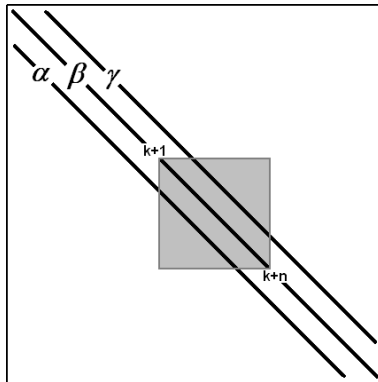
very similar computations show that

$$\text{Spec}_\varepsilon A \subset \overline{\bigcup_{k \in \mathbb{Z}} \text{Spec}_{\varepsilon + \varepsilon'_n} A_{n,k}^{\text{per}}}, \quad \varepsilon \geq 0,$$

$$\text{with} \quad \varepsilon'_n = 2 \sin\left(\frac{\pi}{2n}\right) (\|\alpha\|_\infty + \|\gamma\|_\infty).$$

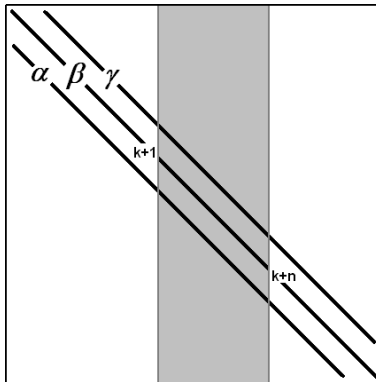
Here is another idea: τ_1 method

Instead of



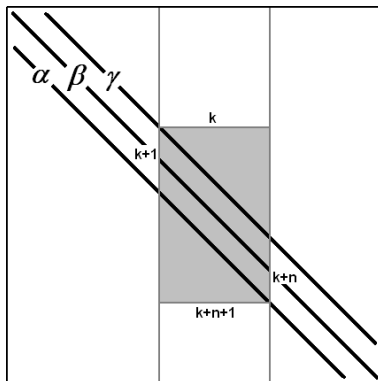
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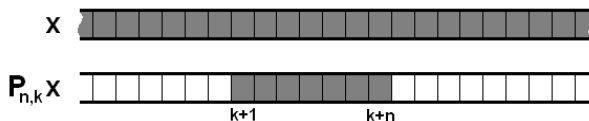


I.e., we work with **rectangular** finite submatrices.

This is motivated by work of Davies 1998, Davies & Plum 2004, and Hansen 2008, 2011, in which A is approximated by a **single large rectangular finite section**.

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let $P_{n,k} : \ell^2 \rightarrow \ell^2$ denote the projection

$$(P_{n,k}x)(i) := \begin{cases} x(i), & i \in \{k+1, \dots, k+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

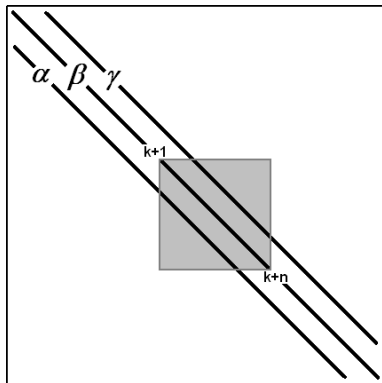


Further, we put

$$E_{n,k} := \text{im } P_{n,k}.$$

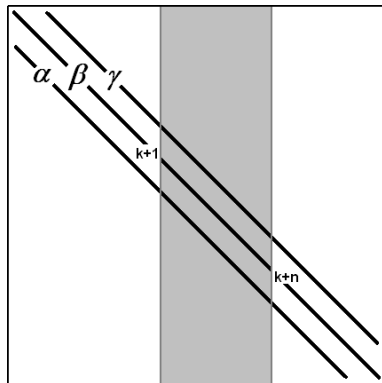
τ_1 method: truncations

τ method:



$$P_{n,k}(A - \lambda I)|_{E_{n,k}}$$

τ_1 method:



$$(A - \lambda I)|_{E_{n,k}}$$

τ method:

$\lambda \in \text{Spec}_\varepsilon A \implies$ For some $k \in \mathbb{Z}$:

$$\lambda \in \text{Spec}_{\varepsilon+\varepsilon_n}(P_{n,k}A|_{E_{n,k}})$$

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$$\text{i.e. } \nu(P_{n,k}(A - \lambda I)|_{E_{n,k}}) \leq \varepsilon + \varepsilon_n \text{ or}$$

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τ_1 idea is just **drop the $P_{n,k}$'s.**

τ_1 method

Let $\gamma_\varepsilon^{n,k}(A)$ be the set of $\lambda \in \mathbb{C}$ for which

$$\min \left(\nu \left((A - \lambda I)|_{E_{n,k}} \right), \nu \left((A - \lambda I)^*|_{E_{n,k}} \right) \right) \leq \varepsilon.$$

(Analogue of $\text{Spec}_\varepsilon A_{n,k}$ in the τ method.)

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Then (similarly to the τ and π -method inclusions)

$$\text{Spec}_\varepsilon A \subset \Gamma_{\varepsilon + \varepsilon_n''}^n(A),$$

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$$\Gamma_\varepsilon^n(A) \subset \text{Spec}_\varepsilon A.$$

τ_1 -method: spectral bounds

From the lower and upper bound

$$\Gamma_{\varepsilon}^n(A) \subset \operatorname{Spec}_{\varepsilon} A \quad \text{and} \quad \operatorname{Spec}_{\varepsilon} A \subset \Gamma_{\varepsilon+\varepsilon_n''}^n(A)$$

we get

Sandwich 1

$$\Gamma_{\varepsilon}^n(A) \subset \operatorname{Spec}_{\varepsilon} A \subset \Gamma_{\varepsilon+\varepsilon_n''}^n(A), \quad \varepsilon \geq 0.$$

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$$\Gamma_{\varepsilon}^n(A) \subset \operatorname{Spec}_{\varepsilon} A \subset \Gamma_{\varepsilon+\varepsilon_n''}^n(A), \quad \varepsilon \geq 0.$$

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τ_1 -method: spectral bounds

From the lower and upper bound

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The shift operator

Let's compute the τ , π , and τ_1 inclusion sets for $\text{Spec } A$, i.e.

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where

$$\gamma_{\epsilon''_n}^{n,k}(A) = \{ \lambda \in \mathbb{C} : \min(\nu((A - \lambda I)|_{E_{n,k}}), \nu((A - \lambda I)^*|_{E_{n,k}})) \leq \epsilon''_n \},$$

in the case that A is the **shift operator**, so that

$$\alpha = (\dots, 0, 0, \dots), \beta = (\dots, 0, 0, \dots), \gamma = (\dots, 1, 1, \dots),$$

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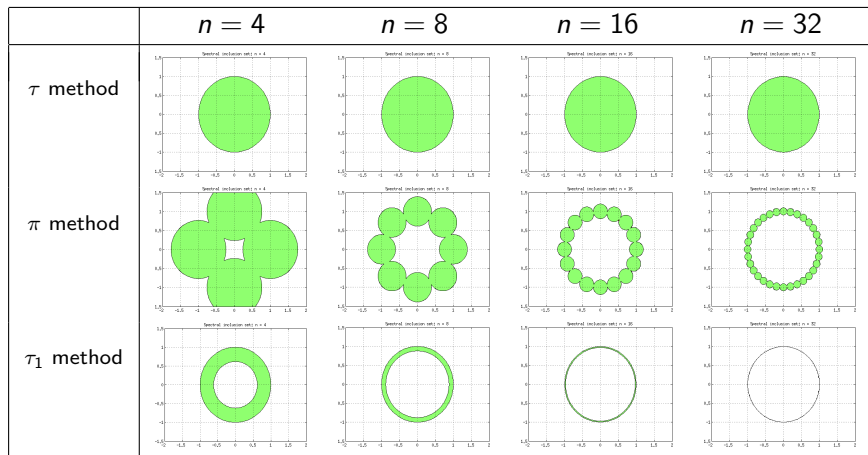
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$$\varepsilon_n, \varepsilon'_n, \varepsilon''_n \leq 2 \sin\left(\frac{\pi}{2n}\right) (\|\alpha\|_\infty + \|\gamma\|_\infty) = 2 \sin\left(\frac{\pi}{2n}\right),$$

and the matrices $A_{n,k}$, $k \in \mathbb{Z}$, are all the same!

The shift operator



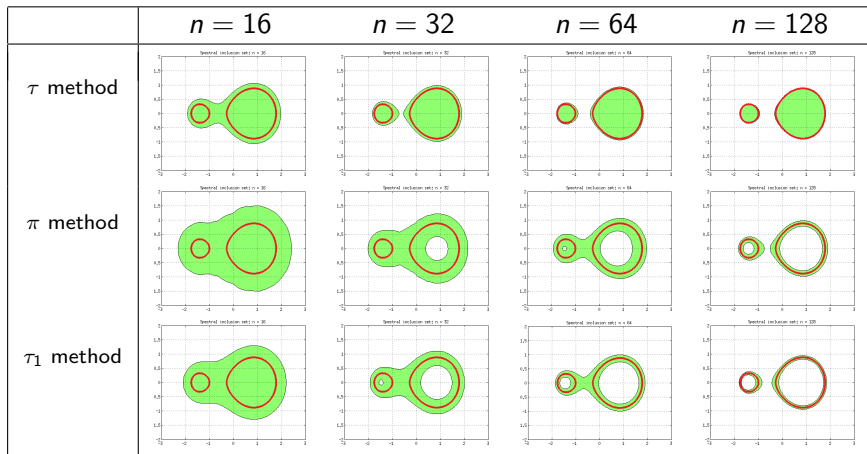
We now look at a tridiagonal matrix A with 3-periodic diagonals:

1st sub-diagonal $\alpha = (\cdots, 0, 0, 0, \cdots)$

main diagonal $\beta = (\cdots, -\frac{3}{2}, 1, 1, \cdots)$

super-diagonal $\gamma = (\cdots, 1, 2, 1, \cdots)$

3-periodic example



Let's take stock: what were we trying to do?

Question. Given a bounded linear operator A on a Hilbert space E , can we construct a sequence of compact sets $U_n \subset \mathbb{C}$ with

- (i) $\text{Spec } A \subset U_n$ for each n ;
- (ii) $U_n \rightarrow \text{Spec } A$ as $n \rightarrow \infty$ (Hausdorff convergence);
- (iii) each U_n can be computed in finitely many operations?

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A final example [Feinberg/Zee 1999]

$$A = \begin{pmatrix} \ddots & \ddots & & & & \\ & \ddots & 0 & 1 & & \\ & & b_{-1} & 0 & 1 & \\ & & & b_0 & 0 & 1 \\ & & & & b_1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where $b = (\cdots, b_{-1}, b_0, b_1, \cdots) \in \{\pm 1\}^{\mathbb{Z}}$ is a **pseudoergodic** sequence (Davies 2001)

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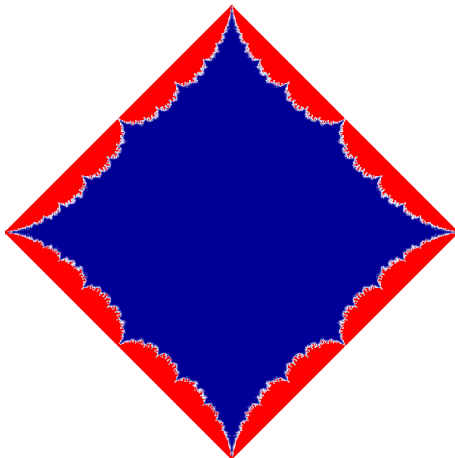
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This is an example where the τ method is convergent:

$$\text{Spec } A \subset U_n := \bigcup_{k \in \mathbb{Z}} \text{Spec}_{\varepsilon_n} A_{n,k} \rightarrow \text{Spec } A, \quad \text{as } n \rightarrow \infty,$$

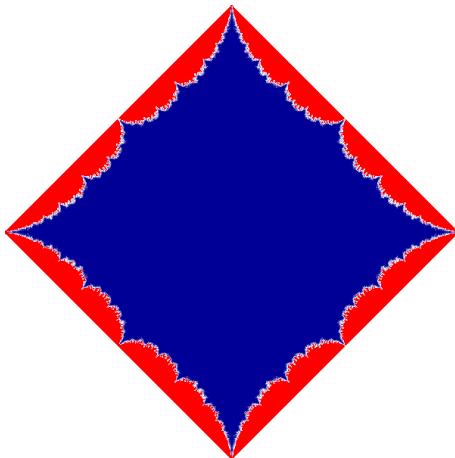
and where the union is finite: 2^{n-1} different matrices $A_{n,k}$.

Upper and lower bounds on $\text{Spec } A$: which is sharp?



(The square has corners at ± 2 and $\pm 2i$.)

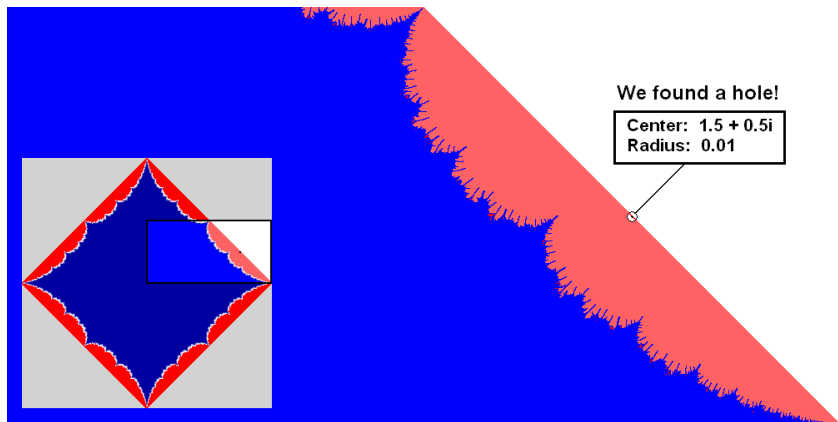
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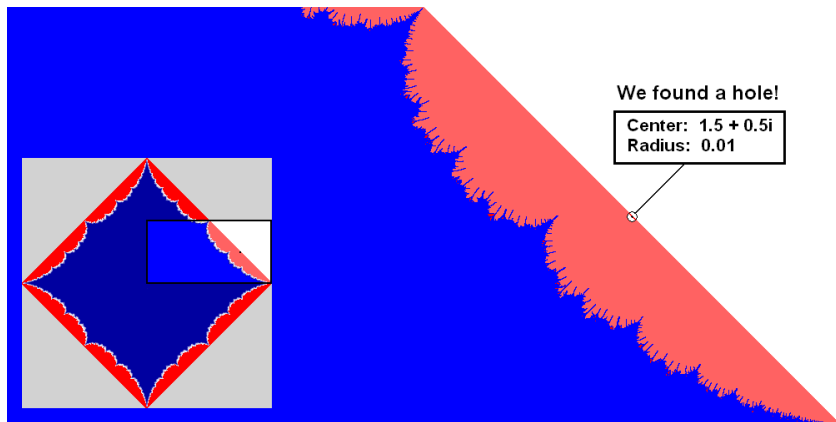
We have $\text{Spec } A \subset U_n$ and $U_n \rightarrow \text{Spec } A$ so, if $\lambda \notin \text{Spec } A$, then $\lambda \notin U_n$ **for all sufficiently large n .**

Is $\lambda = 1.5 + 0.5i \in \text{Spec } A$?



$\lambda = 1.5 + 0.5i \notin U_{34} \supset \text{Spec } A$, so $\lambda \notin \text{Spec } A$,
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so $\text{Spec } A$ is a strict subset of the square. This was a large calculation: we needed to check whether $2^{33} \approx 8.6 \times 10^9$ matrices of size 34×34 were positive definite!

Summary and conclusion

1. For **tridiagonal** A we have derived the τ , π , and τ_1 **inclusion set families** for $\text{Spec}_\varepsilon A$, for $\varepsilon \geq 0$, i.e., for $n \in \mathbb{N}$,

$$\tau \text{ method:} \quad \text{Spec}_\varepsilon A \subset \overline{\bigcup_{k \in \mathbb{Z}} \text{Spec}_{\varepsilon + \varepsilon_n} A_{n,k}}$$

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4. Sketched extension to A band-dominated, and how τ_1 -method can be adapted to need only finitely many operations while maintaining inclusion and convergence properties. 

On Spectral Inclusion Sets and Computing the Spectra and Pseudospectra of Bounded Linear Operators

SIMON N. CHANDLER-WILDE, RATCHANIKORN CHONCHAIYA,
and MARKO LINDNER

January 11, 2024

ABSTRACT. In this paper we derive novel families of inclusion sets for the spectrum and pseudospectrum of large classes of bounded linear operators, and establish convergence of particular sequences of these inclusion sets to the spectrum or pseudospectrum, as appropriate. Our results apply, in particular, to bounded linear operators on a separable Hilbert space that, with respect to some orthonormal basis, have a representation as a bi-infinite matrix that is banded or band-dominated. More generally, our results apply in cases where the matrix entries themselves are bounded linear operators on some Banach space. In the scalar matrix entry case we show that our methods, given the input information we assume, lead to a sequence of approximations to the spectrum, each element of which can be computed in finitely many arithmetic operations, so that, with our assumed inputs, the problem of determining the spectrum of a band-dominated operator has solvability complexity index one, in the sense of Ben-Artzi et al. (*C. R. Acad. Sci. Paris, Ser. I* **353** (2015), 931–936). As a concrete and substantial application, we apply our methods to the determination of the spectra of non-self-adjoint bi-infinite tridiagonal matrices that are pseudoergodic in the sense of Davies (*Commun. Math. Phys.* **216** (2001) 687–704).

Mathematics subject classification (2010): Primary 47A10; Secondary 47B36, 46E40, 47B80.

Keywords: band matrix, band-dominated matrix, solvability complexity index, pseudoergodic

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